J. Phys. A: Math. Gen. **31** (1998) 2109–2121. Printed in the UK PII: S0305-4470(98)88802-1

Complete non-diagonal reflection matrices of RSOS/SOS and hard hexagon models

Changrim Ahn and Choong-Ki You
Department of Physics, Ewha Womans University, Seoul 120-750, Korea

Received 28 October 1997

Abstract. In this paper we compute the most general non-diagonal reflection matrices of the RSOS/SOS models and the hard hexagon model using the boundary Yang–Baxter equations. We find a new one-parameter family of the reflection matrices for the RSOS model in addition to the previous result obtained in (Ahn C and Koo W M 1996 *Nucl. Phys.* B **468** [FS] 461). We also find three classes of the reflection matrices for the SOS model, which has one or two free parameters. For the hard-hexagon model which can be mapped to the RSOS(5) model by folding four RSOS heights into two, the solutions can be obtained similarly with a main difference in the boundary unitarity conditions. Due to this, the reflection matrices can have two free parameters. We show that these extra terms can be identified with the 'decorated' solutions. We also generalize the hard hexagon model by 'folding' the RSOS heights of the general RSOS(p) model and show that they satisfy the integrability conditions such as the Yang–Baxter and boundary Yang–Baxter equations. These models can be solved using the results for the RSOS models.

1. Introduction

In the study of the two-dimensional integrable models of quantum field theories and statistical models, the Yang–Baxter equation (YBE) plays an essential role in estabilishing the integrability and solving the models. Recently much effort has been directed to introducing boundaries into the integrable systems for possible application to condensed-matter physics and statistical systems with non-periodic boundary conditions. The boundaries entail new a physical quantity called reflection matrices which depend on the boundary properties.

The boundary Yang–Baxter equation (BYBE) (also known as the reflection equation) [1] is the necessary condition for integrable statistical models [2, 3] and quantum field theories [4] with a boundary. The equation takes the form

$$R_1(u)S_{12}(u'+u)R_2(u')S_{12}(u'-u) = S_{12}(u'-u)R_2(u')S_{12}(u'+u)R_1(u) \tag{1.1}$$

where $R_{1(2)}$ is the boundary reflection matrix in the auxiliary space 1(2) and S_{12} is the solution to the YBE.

To date, several solutions of the BYBE have appeared in the literature. Compared with the vertex-type models, however, far less is known about the solution in the face-type model, such as the solid-on-solid (SOS) or restricted-solid-on-solid (RSOS) model. Among these, most of the known solutions are 'diagonal' in the sense that the reflection matrices are diagonal [5, 6]. Using these, one can find the partition functions in the infinite lattice limit [7]. In addition, diagonal reflection matrices have recently been applied to interesting phenomena like condensed-matter physics.

Non-diagonal solutions show more non-trivial boundary scattering and may also be useful for physical applications. Furthermore, one can find an explicit vertex–face correspondence if one can classify non-diagonal solutions completely. In addition to those found in [6], we will show that different classes of non-diagonal reflection matrices are possible. These new solutions include one or two free parameters which are related to those in the boundary potential. The explicit formulae between two sets of parameters are still not clear. We hope our complete non-diagonal solutions for the SOS model may be related explicitly with those of the boundary sine-Gordon model.

From a physical point of view, many interesting models are of face type: the RSOS/SOS models, the hard-hexagon model (HHM), etc. These models play very important roles in statistical mechanics systems and in quantum field theories such as the perturbed conformal field theories.

In this paper we derive complete non-diagonal reflection matrices for the RSOS/SOS and the HHM in a unified way. We will express the BYBE as a linear equation which should satisfy extra non-trivial conditions. Due to the linearity, the general solutions are linear combinations of each solution with arbitrary coefficients. Some of these coefficients are fixed by the boundary crossing and unitarity conditions [4]. The reflection matrices of the HHM can be constructed from those of the RSOS model since the HHM can be mapped to the RSOS(5). We consider a similar mapping for the generic RSOS(p) model and its boundary reflection matrices.

2. The RSOS(p) model

In this section we solve the BYBE for the RSOS(p); $p = 3, 4 \ldots$ scattering theory. The RSOS(p) scattering theory is based on a $(p-1)$-fold degenerate vacuum structure, in which vacua can be associated with nodes of the $\mathcal{A}_{p-1}$ Dynkin diagram. The quasiparticles in the scattering theory are kinks that interpolate neighbouring vacua, they can be denoted by non-commutative symbols $K_{ab}(u)$ where $|a-b| = 1$ with $a, b = 1, \ldots, p-1$ and u is related to the the kink rapidity θ by $u = -\mathrm{i}\theta/p$, so that the physical strip is given by $0 < \mathrm{Re}\, u < \pi/p$. In the rest of the paper, we will refer to a, b as heights or spins. Formally, scattering between two kinks can be represented by the following equation:

$$K_{da}(u)K_{ab}(u') = \sum_c S^{ab}_{dc}(u-u')K_{dc}(u')K_{cb}(u) \tag{2.1}$$

where the S-matrix is given by

$$S^{ab}_{dc}(u) = \mathcal{U}(u)\left(\frac{[a][c]}{[d][b]}\right)^{-u/2\gamma} W^{ab}_{dc}(u). \tag{2.2}$$

The Boltzmann weight

$$W^{ab}_{dc}(u) = \sin u\,\delta_{bd}\left(\frac{[a][c]}{[d][b]}\right)^{1/2} + \sin(\gamma - u)\delta_{ac} \tag{2.3}$$

satisfies the YBE in the RSOS representation.

Here $[a]$ denotes the usual q-number

$$[a] = \frac{\sin(a\gamma)}{\sin\gamma} \qquad \gamma = \frac{\pi}{p}$$

and the overall factor $\mathcal{U}(u)$ is a product of Gamma functions

$$\mathcal{U}(u) = \frac{1}{\pi}\Gamma\left(\frac{\gamma}{\pi}\right)\Gamma\left(1-\frac{u}{\pi}\right)\Gamma\left(1-\frac{\gamma}{\pi}+\frac{u}{\pi}\right)\prod_{l=1}^{\infty}\frac{F_l(u)F_l(\gamma-u)}{F_l(0)F_l(\gamma)}$$
$$F_l(u) = \frac{\Gamma\left(2l\gamma/\pi - u/\pi\right)\Gamma\left(1+2l\gamma/\pi-u/\pi\right)}{\Gamma\left((2l+1)\gamma/\pi-u/\pi\right)\Gamma\left(1+(2l-1)\gamma/\pi-u/\pi\right)}. \tag{2.4}$$

This factor, satisfying the relations

$$\mathcal{U}(u)\mathcal{U}(-u)\sin(\gamma-u)\sin(\gamma+u) = 1$$

$$\mathcal{U}(\gamma-u) = \mathcal{U}(u)$$

together with the overall q-number factor ensures that the S-matrix satisfies both the crossing and unitarity constraints:

$$S^{bc}_{ad}(\gamma-u) = S^{ab}_{dc}(u) \tag{2.5}$$

$$\sum_{c'} S^{ab}_{dc'}(u)S^{c'b}_{dc}(-u) = \delta_{ac}. \tag{2.6}$$

Let us now consider the above scattering theory in the presence of a boundary. The scattering between the kink and the boundary denoted formally by $\mathbf{B}_a$ is described by the equation

$$K_{ab}(u)\mathbf{B}_a = \sum_c R^b_{ac}(u)K_{bc}(-u)\mathbf{B}_c. \tag{2.7}$$

Note that in this representation, the boundary naturally carries an RSOS spin.

The function R^b_{ac} is called the boundary reflection matrix and satisfies the BYBE, which in the RSOS representation takes the form

$$\sum_{a',b'} R^a_{bb'}(u)S^{ac}_{b'a'}(u'+u)R^{a'}_{b'b''}(u')S^{a'c}_{b''a''}(u'-u)$$
$$= \sum_{a',b'} S^{ac}_{ba'}(u'-u)R^{a'}_{bb'}(u')S^{a'c}_{b'a''}(u'+u)R^{a''}_{b'b''}(u). \tag{2.8}$$

In general, the function $R^a_{bc}(u)$ can be written as

$$R^a_{bc}(u) = \mathcal{R}(u)\left(\frac{[b][c]}{[a][a]}\right)^{-u/2\gamma}\left[\delta_{b\neq c}X^a_{bc}(u) + \delta_{bc}\left\{\delta_{b,a+1}U_a(u) + \delta_{b,a-1}D_a(u)\right\}\right] \tag{2.9}$$

where $\mathcal{R}(u)$ has to be determined from the boundary crossing and unitarity constraints, while X^a_{bc} and U_a, D_a have to be determined from the BYBE. We have multiplied a prefactor depending on q-numbers to simplify the BYBE. If X^a_{bc} does not vanish, the boundary R-matrix describes non-diagonal scattering process, otherwise the scattering is called diagonal. Note that due to the restriction that the vacuum assumes the value $1, \ldots, p-1$, $X^1_{bc}, X^{p-1}_{bc}, D_1, U_{p-1}$ are not defined. The case $p = 3$ has only diagonal scattering, so X^a_{bc} does not exist.

We will concentrate in what follows on the scattering where the off-diagonal component X^a_{bc} is non-vanishing. To start, the case $b \neq c \neq b''$ in equation (2.8) gives

$$X^a_{a-1,a+1}(u')X^{a+2}_{a+1,a+3}(u) = X^a_{a-1,a+1}(u)X^{a+2}_{a+1,a+3}(u') \qquad 2 \leqslant a \leqslant p-4. \tag{2.10}$$

This equation implies that $X^a_{a\pm1,a\mp1}$ can be written as

$$X^a_{a\pm1,a\mp1}(u) = h_\pm(u)X^a_\pm$$

where $h_\pm(u)$ depends only on u and $X^a_\pm$ only on a.

On the other hand, the case $c = b = b''$, $a = a''$ gives

$$X^a_{a-1,a+1}(u')X^a_{a+1,a-1}(u) = X^a_{a-1,a+1}(u)X^a_{a+1,a-1}(u') \qquad 2 \leqslant a \leqslant p-2 \tag{2.11}$$

which implies that

$$h_+(u')h_-(u) = h_+(u)h_-(u')$$

from which we conclude that

$$h_+(u) = (\text{constant})h_-(u).$$

Absorbing the constant in the above equation into X^a_- or X^a_+, we can make h_+ equal to h_- so that we can absorb the $h_\pm(u)$ into the overall $\mathcal{R}(u)$ factor and treat X^a_{bc} as u independent from now on.

With this simplification, equation (2.8) can be broken down into the following independent equations in addition to the above two equations:

$$\begin{aligned} U'_a D_{a+2} f_+ \left(1 + f_- \frac{[a]}{[a+1]}\right) &+ D'_{a+2} D_{a+2} f_- \left(1 + f_+ \frac{[a+2]}{[a+1]}\right) \\ &+ X^{a+2}_{a+1,a+3} X^{a+2}_{a+3,a+1} f_- = U_a D'_{a+2} f_+ \left(1 + f_- \frac{[a+2]}{[a+1]}\right) \\ &+ U'_a U_a f_- \left(1 + f_+ \frac{[a]}{[a+1]}\right) + X^a_{a-1,a+1} X^a_{a+1,a-1} f_- \end{aligned} \tag{2.12}$$

for $1 \leqslant a \leqslant p-3$,

$$D'_{a+1} f_- \left(1 + f_+ \frac{[a+1]}{[a]}\right) + U'_{a-1} f_+ \left(1 + f_- \frac{[a-1]}{[a]}\right) = U_{a-1} f_+ - U_{a+1} f_- \tag{2.13}$$

$$U'_a f_- \left(1 + f_+ \frac{[a]}{[a+1]}\right) + D'_{a+2} f_+ \left(1 + f_- \frac{[a+2]}{[a+1]}\right) = D_{a+2} f_+ - D_a f_- \tag{2.14}$$

for $2 \leqslant a \leqslant p-3$, and

$$\begin{aligned} U'_{a-2} f_+ f_- \frac{[a][a-2]}{[a-1]^2} - U'_a + D'_a \left(1 + f_- \frac{[a]}{[a-1]}\right)\left(1 + f_+ \frac{[a]}{[a-1]}\right) \\ = D_a \left(1 + f_+ \frac{[a]}{[a-1]}\right) - U_a \left(1 + f_- \frac{[a]}{[a-1]}\right) \end{aligned} \tag{2.15}$$

$$\begin{aligned} D'_{a+2} f_+ f_- \frac{[a][a+2]}{[a+1]^2} - D'_a + U'_a \left(1 + f_- \frac{[a]}{[a+1]}\right)\left(1 + f_+ \frac{[a]}{[a+1]}\right) \\ = U_a \left(1 + f_+ \frac{[a]}{[a+1]}\right) - D_a \left(1 + f_- \frac{[a]}{[a+1]}\right) \end{aligned} \tag{2.16}$$

for $2 \leqslant a \leqslant p-2$. In the above equations, we used a compact notation where $U_a = U_a(u)$, $U'_a = U_a(u')$ (similarly for D_a) and

$$f_\pm = \sin(u' \pm u)/\sin(\gamma - u' \mp u).$$

In addition, it should also be mentioned that the last term on the right-hand side (left-hand side) of equation (2.12) is present only when $a \neq 1$ $(p-3)$ and the first terms of equations (2.15) and (2.16) are allowed only for $a \neq 2$ and $a \neq p-2$, respectively.

Before solving the equations directly, it is helpful to investgate the structure of the BYBE. First of all, the BYBE is covariant under the transformation

$$a \to p - a \qquad U_a(u) \to \pm D_{p-a}(u) \tag{2.17}$$

and has the following symmetry for $2 \leqslant a \leqslant p - 2$:

$$U_a(u) = -D_a(-u). \tag{2.18}$$

Futhermore, since equations (2.13)–(2.16) are linear, their general solutions are linear combinations of the 'fundamental' ones. Another important fact is that the amplitudes with a even and those with a odd are completely decoupled in the BYBE and give different solutions in general. However, if p is odd, two sets are related by (2.17) and have the same solutions.

By solving the linear equations (2.13)–(2.16), we find the most general non-diagonal solution for $p \geqslant 5$:

$$\begin{aligned} U_a(u) &= A\sin(2u + a\gamma) + \frac{B}{\sin 2u} + \frac{\epsilon_p\epsilon_{a-1}C}{\sin a\gamma} + \frac{\epsilon_{p-1}D}{\sin a\gamma}\left\{\frac{\sin(2u + a\gamma)}{\sin 2u} - (-1)^a\right\} \\ D_a(u) &= A\sin(2u - a\gamma) + \frac{B}{\sin 2u} - \frac{\epsilon_p\epsilon_{a-1}C}{\sin a\gamma} - \frac{\epsilon_{p-1}D}{\sin a\gamma}\left\{\frac{\sin(2u - a\gamma)}{\sin 2u} - (-1)^a\right\} \end{aligned} \tag{2.19}$$

where ϵ_a is 0 (1) if a is odd (even) and $A,\ B,\ C$ and D are free parameters.

Having found U_a, D_a, the function X^a_{bc} can be easily obtained from equation (2.12), after taking u' to be $-u$ since X^a_{bc} does not depend on the rapidity. This gives

$$X^{a+2}_{a+1,a+3}X^{a+2}_{a+3,a+1} - X^a_{a-1,a+1}X^a_{a+1,a-1} = U_a(-u)U_a(u) - D_{a+2}(-u)D_{a+2}(u)$$

for $1 \leqslant a \leqslant p - 3$. Substituting U_a, D_a in the right-hand side and iterating the equations, we get

$$\begin{aligned} X^a_{a-1,a+1}X^a_{a+1,a-1} &= \epsilon_p\epsilon_a X^2_{13}X^2_{31} + A^2\left\{\sin^2(1 + \epsilon_p\epsilon_a)\gamma - \sin^2 a\gamma\right\} \\ &\quad - 2AB\left\{\cos(1 + \epsilon_p\epsilon_a)\gamma - \cos a\gamma\right\} + \epsilon_p\epsilon_{a-1}C^2\left\{\frac{1}{\sin^2\gamma} - \frac{1}{\sin^2 a\gamma}\right\} \\ &\quad + 2\epsilon_{p-1}D^2\left\{\frac{1}{\sin^2\gamma} - \frac{1}{\sin^2 a\gamma}\right\} + 2\epsilon_{p-1}D^2\left\{\frac{\cos\gamma}{\sin^2\gamma} + (-1)^a\frac{\cos a\gamma}{\sin^2 a\gamma}\right\}. \end{aligned} \tag{2.20}$$

Since this equation fixes only the product, $X^a_{a-1,a+1}$ and $X^a_{a+1,a-1}$ are determined up to a gauge factor. Inserting $a = 1$ one gets $X^1_{02}X^1_{20} = 0$ as expected. For p even, $X^2_{13}X^2_{31}$ is not yet determined.

Now consider the boundary unitarity and crossing symmetry conditions for the reflection matrix $R^a_{bc}(u)$. Due to these conditions, the overall factor $R(u)$ should satisfy

$$\sum_c R^a_{bc}(u)R^a_{cd}(-u) = \delta_{bd} \tag{2.21}$$

$$\sum_d S^{ac}_{bd}(2u)R^d_{bc}(\gamma/2 + u) = R^a_{bc}(\gamma/2 - u). \tag{2.22}$$

In terms of equations (2.9) and (2.18), the unitarity condition becomes

$$\mathcal{R}(u)\mathcal{R}(-u)\left[X^a_{a+1,a-1}X^a_{a-1,a+1} - U_a(u)D_a(u)\right] = 1 \qquad 2 \leqslant a \leqslant p - 2$$

$$\mathcal{R}(u)\mathcal{R}(-u)U_1(u)U_1(-u) = 1$$

$$\mathcal{R}(u)\mathcal{R}(-u)D_{p-1}(u)D_{p-1}(-u) = 1.$$

By inserting equations (2.19) and (2.20) in the above, we find the following non-trivial constraints for the free parameters:

(i) $AB = AC = AD = 0$

(ii) $X_{13}^2 X_{31}^2 = A^2(\sin^2\gamma - \sin^2 2\gamma) + \dfrac{C^2}{\sin^2\gamma}$ for even p.

Note that all the cases satisfying the constraints, have at most one free parameter since we can absorb overall constant into $\mathcal{R}(u)$. We list each class of the reflection matrices as follows.

Class I. For general p $(p \neq 3, 4)$: $B = C = D = 0$, $A = 1$

$$\begin{aligned} U_a(u) &= \sin(2u + a\gamma) \\ D_a(u) &= \sin(2u - a\gamma) \\ X_{a-1,a+1}^a X_{a+1,a-1}^a &= \sin^2\gamma - \sin^2 a\gamma. \end{aligned} \tag{2.23}$$

These weights have $U_{p-a}(u) = -D_a(u)$ symmetry and no free parameter. This solution is the one obtained in [6]. The unitarity condition gives

$$\mathcal{R}(u)\mathcal{R}(-u)(-\sin^2 2u + \sin^2\gamma) = 1. \tag{2.24}$$

The crossing symmetry condition becomes

$$\mathcal{U}(2u)\mathcal{R}(\gamma/2 + u)\sin(\gamma - 2u) = \mathcal{R}(\gamma/2 - u). \tag{2.25}$$

The factor $\mathcal{R}(u)$ can be determined from equations (2.24), (2.25) up to the usual CDD ambiguity by separating $\mathcal{R}(u) = \mathcal{R}_0(u)\mathcal{R}_1(u)$ where $\mathcal{R}_0$ satisfies

$$\begin{aligned} &\mathcal{R}_0(u)\mathcal{R}_0(-u) = 1 \\ &\mathcal{U}(2u)\mathcal{R}_0(\gamma/2 + u)\sin(\gamma - 2u) = \mathcal{R}_0(\gamma/2 - u) \end{aligned} \tag{2.26}$$

whose minimal solution reads

$$\mathcal{R}_0(u) = \frac{F_0(u)}{F_0(-u)}.$$

$\mathcal{R}_1$ satisfies

$$\begin{aligned} &\mathcal{R}_1(u)\mathcal{R}_1(-u)(-\sin^2 2u + \sin^2\gamma) = 1 \\ &\mathcal{R}_1(u) = \mathcal{R}_1(\gamma - u) \end{aligned} \tag{2.27}$$

with the minimal solution

$$\mathcal{R}_1(u) = \tfrac{1}{2}\sigma(\gamma/2, u)\sigma(\pi/2 - \gamma/2, u).$$

Here $\sigma(x, u)$ is a well known building block satisfying the relations

$$\sigma(x, u) = \sigma(x, \gamma - u)$$

$$\sigma(x, u)\sigma(x, -u) = [\cos(x + u)\cos(x - u)]^{-1}$$

and is given by

$$\sigma(x, u) = \frac{\prod(x, \gamma/2 - u)\prod(-x, \gamma/2 - u)\prod(x, -\gamma/2 + u)\prod(-x, -\gamma/2 + u)}{\prod^2(x, \gamma/2)\prod^2(-x, \gamma/2)}$$

$$\prod(x, u) = \prod_{l=0}^{\infty} \frac{\Gamma\left(\frac{1}{2} + (2l + \frac{1}{2})\gamma/\pi + x/\pi - u/\pi\right)}{\Gamma\left(\frac{1}{2} + (2l + \frac{3}{2})\gamma/\pi + x/\pi - u/\pi\right)}.$$

Class II. For even p $(p \neq 4)$: $A = 0,\ C = 1$

$$
\begin{aligned}
U_a(u) &= \frac{B}{\sin 2u} + \frac{\epsilon_{a-1}}{\sin a\gamma} \\
D_a(u) &= \frac{B}{\sin 2u} - \frac{\epsilon_{a-1}}{\sin a\gamma} \\
X^a_{a-1,a+1}X^a_{a+1,a-1} &= \frac{1}{\sin^2\gamma} - \frac{\epsilon_{a-1}}{\sin^2 a\gamma}.
\end{aligned}
\tag{2.28}
$$

This solution satisfies $U_{p-a}(u) = U_p(u)$ (similarly for $D_a(u)$ and X^a_{bc}) and includes one free parameter. To fix the overall factor $\mathcal{R}(u)$, $\mathcal{R}_0(u)$ is the same as for class I, while $\mathcal{R}_1(u)$ satisfies

$$
\mathcal{R}_1(u)\mathcal{R}_1(-u)\left(\frac{1}{\sin^2\gamma} - \frac{B^2}{\sin^2 2u}\right) = 1. \tag{2.29}
$$

The minimal solution is

$$
\mathcal{R}_1(u) = \sin\gamma\ \frac{\sigma(x,u)\sigma(\pi/2-x,u)}{\sigma(0,u)\sigma(\pi/2,u)}
$$

where

$$
\sin 2x = B\sin\gamma.
$$

Class III. For odd p $(p \neq 3)$: $A = 0,\ D = 1$

$$
\begin{aligned}
U_a &= \frac{B}{\sin 2u} + \frac{1}{\sin a\gamma}\left\{\frac{\sin(2u+a\gamma)}{\sin 2u} - (-1)^a\right\} \\
D_a &= \frac{B}{\sin 2u} - \frac{1}{\sin a\gamma}\left\{\frac{\sin(2u-a\gamma)}{\sin 2u} - (-1)^a\right\} \\
X^a_{a-1,a+1}X^a_{a+1,a-1} &= 2\left\{\frac{1}{\sin^2\gamma} - \frac{1}{\sin^2 a\gamma}\right\} + 2\left\{\frac{\cos\gamma}{\sin^2\gamma} + (-1)^a\frac{\cos a\gamma}{\sin^2 a\gamma}\right\}.
\end{aligned}
\tag{2.30}
$$

This solution has $U_{p-a}(u) = D_p(u)$ symmetry and one free parameter. While $\mathcal{R}_0(u)$ does not change, the $\mathcal{R}_1(u)$ satisfies

$$
\mathcal{R}_1(u)\mathcal{R}_1(-u)\left(\frac{1}{\sin^2(\gamma/2)} - \frac{2B\cos 2u}{\sin^2 2u} - \frac{1+B^2}{\sin^2 2u}\right) = 1.
$$

The minimal solution is

$$
\mathcal{R}_1(u) = \sin\frac{\gamma}{2}\ \frac{\sigma(x_1,u)\sigma(x_2,u)}{\sigma(0,u)\sigma(\pi/2,u)}
$$

where

$$
\cos^2 x_1 + \cos^2 x_2 = 1 + B\sin^2\frac{\gamma}{2}
$$

$$
\cos x_1\cos x_2 = \frac{1}{2}(1+B)\sin\frac{\gamma}{2}.
$$

In the above analysis, we omit special cases of $p = 3, 4$ since $p = 3$ has only the diagonal reflection matrices and $p = 4$ has been extensively studied in [8].

3. The SOS model

In the beginning we have considered that the heights take values from 1 to $p-1$, which is necessary for the bulk scattering weights to be finite, as the parameter $\pi/\gamma = p$ is a positive integer. When π/γ is not a rational number, there are no bounds on the heights and the corresponding representation is known as solid-on-solid (SOS). Since there is no restriction on the heights, we set $\epsilon_p = 1$ in the solutions (2.19) of the BYBE. Thus the solutions of BYBE in the SOS representation are

$$
\begin{aligned}
U_a(u) &= A\sin(2u+a\gamma) + \frac{B}{\sin 2u} + \frac{\epsilon_{a-1}C}{\sin a\gamma} + \frac{D}{\sin a\gamma}\left\{\frac{\sin(2u+a\gamma)}{\sin 2u} - 1\right\} \\
D_a(u) &= A\sin(2u-a\gamma) + \frac{B}{\sin 2u} - \frac{\epsilon_{a-1}C}{\sin a\gamma} - \frac{D}{\sin a\gamma}\left\{\frac{\sin(2u-a\gamma)}{\sin 2u} - 1\right\} \\
X^a_{a-1,a+1}X^a_{a+1,a-1} &= \epsilon_a X^2_{13}X^2_{31} + A^2\left\{\sin^2(1+\epsilon_a)\gamma - \sin^2 a\gamma\right\} \\
&\quad - 2AB\left\{\cos(1+\epsilon_a)\gamma - \cos a\gamma\right\} + \epsilon_{a-1}C^2\left\{\frac{1}{\sin^2\gamma} - \frac{1}{\sin^2 a\gamma}\right\} \\
&\quad - \epsilon_{a-1}CD\left\{\frac{1}{\cos^2(\gamma/2)} - \frac{1}{\cos^2(a\gamma/2)}\right\} \\
&\quad + D^2\left\{\frac{1}{\cos^2((1+\epsilon_a)\gamma/2)} - \frac{1}{\cos^2(a\gamma/2)}\right\}
\end{aligned}
\tag{3.1}
$$

redefining C as $C-2D$.

Inserting the above solution in the unitarity condition restricts the coefficients in the same way as in the RSOS(p). We classify the solutions into three classes.

Class I. $C=0,\ A=1$

$$
\begin{aligned}
U_a(u) &= \sin(2u+a\gamma) + \frac{B}{\sin 2u} + \frac{D}{\sin a\gamma}\left\{\frac{\sin(2u+a\gamma)}{\sin 2u} - 1\right\} \\
D_a(u) &= \sin(2u-a\gamma) + \frac{B}{\sin 2u} - \frac{D}{\sin a\gamma}\left\{\frac{\sin(2u-a\gamma)}{\sin 2u} - 1\right\} \\
X^a_{a-1,a+1}X^a_{a+1,a-1} &= \sin^2\gamma - \sin^2 a\gamma - 2B(\cos\gamma - \cos a\gamma) \\
&\quad + D^2\left(\frac{1}{\cos^2(\gamma/2)} - \frac{1}{\cos^2(a\gamma/2)}\right).
\end{aligned}
\tag{3.2}
$$

The overall factor $\mathcal{R}_0$ is the same as that for RSOS(p), but $\mathcal{R}_1(u)$ now contains all the information about the boundary conditions and has to satisfy

$$
\begin{aligned}
\mathcal{R}_1(u)\mathcal{R}_1(-u)\bigg(&-\sin^2 2u - 2D\cos 2u + \sin^2\gamma - 2B\cos\gamma + \frac{D^2}{\cos^2(\gamma/2)} \\
&- \frac{2BD\cos 2u}{\sin^2 2u} - \frac{B^2+D^2}{\sin^2 2u}\bigg) = 1 \\
\mathcal{R}_1(u) &= \mathcal{R}_1(\gamma-u)
\end{aligned}
\tag{3.3}
$$

whose minimal solution is

$$
\mathcal{R}_1(u) = \frac{\sigma(x_1,u)\sigma(x_2,u)\sigma(x_3,u)\sigma(x_4,u)}{2\sigma(0,u)\sigma(\pi/2,u)}
$$

where $x_1, \ldots, x_4$ are related to B, D via

$$\sum_{i=1}^{4} \cos 2x_i = -2D$$

$$\sum_{i>j=1}^{4} \cos 2x_i \cos 2x_j = -2 + \sin^2\gamma - 2B\cos\gamma + \frac{D^2}{\cos^2(\gamma/2)}$$

$$\sum_{i>j>k=1}^{4} \cos 2x_i \cos 2x_j \cos 2x_k = 2(1+B)D$$

$$\cos 2x_1 \cos 2x_2 \cos 2x_3 \cos 2x_4 = (\cos\gamma + B)^2 + D^2 \tan^2 \frac{\gamma}{2}.$$

Class II. $A = 0,\ C = 1$

$$U_a(u) = \frac{B}{\sin 2u} + \frac{\epsilon_{a-1}}{\sin a\gamma} + \frac{D}{\sin a\gamma}\left\{\frac{\sin(2u + a\gamma)}{\sin 2u} - 1\right\}$$

$$D_a(u) = \frac{B}{\sin 2u} - \frac{\epsilon_{a-1}}{\sin a\gamma} - \frac{D}{\sin a\gamma}\left\{\frac{\sin(2u - a\gamma)}{\sin 2u} - 1\right\} \tag{3.4}$$

$$X^a_{a-1,a+1} X^a_{a+1,a-1} = \frac{\{D(1-\cos\gamma) - 1\}^2}{\sin^2\gamma} - \frac{\{D(1-\cos a\gamma) - \epsilon_{a-1}\}^2}{\sin^2 a\gamma}.$$

Now

$$\mathcal{R}_1(u)\mathcal{R}_1(-u)\left[\frac{\{D(1-\cos\gamma)-1\}^2}{\sin^2\gamma} + D^2 - \frac{2BD\cos 2u}{\sin^2 2u} - \frac{B^2 + D^2}{\sin^2 2u}\right] = 1$$

whose minimal solution is

$$\mathcal{R}_1(u) = \frac{\sin\gamma}{\sqrt{\{D(1-\cos\gamma)-1\}^2 + D^2\sin^2\gamma}}\ \frac{\sigma(x_1,u)\sigma(x_2,u)}{\sigma(0,u)\sigma(\pi/2,u)}$$

where x_1, x_2 are related to B, D via

$$\cos x_1 \cos x_2 = \frac{(B+D)\sin\gamma}{2\sqrt{\{D(1-\cos\gamma)-1\}^2 + D^2\sin^2\gamma}}$$

$$\cos^2 x_1 + \cos^2 x_2 = 1 + \frac{BD\sin\gamma}{\{D(1-\cos\gamma)-1\}^2 + D^2\sin^2\gamma}\ .$$

Class III. $A = C = 0,\ D = 1$

$$U_a(u) = \frac{B}{\sin 2u} + \frac{1}{\sin a\gamma}\left\{\frac{\sin(2u + a\gamma)}{\sin 2u} - 1\right\}$$

$$D_a(u) = \frac{B}{\sin 2u} - \frac{1}{\sin a\gamma}\left\{\frac{\sin(2u - a\gamma)}{\sin 2u} - 1\right\} \tag{3.5}$$

$$X^a_{a-1,a+1} X^a_{a+1,a-1} = \frac{1}{\cos^2(\gamma/2)} - \frac{1}{\cos^2(a\gamma/2)}.$$

Now

$$\mathcal{R}_1(u)\mathcal{R}_1(-u)\left(\frac{1}{\cos^2(\gamma/2)} - \frac{2B\cos 2u}{\sin^2 2u} - \frac{1+B^2}{\sin^2 2u}\right) = 1$$

with the minimal solution

$$\mathcal{R}_1(u) = \cos\frac{\gamma}{2}\;\frac{\sigma(x_1,u)\sigma(x_2,u)}{\sigma(0,u)\sigma(\pi/2,u)}$$

where

$$\cos^2 x_1 + \cos^2 x_2 = 1 + B\cos^2(\gamma/2)$$

$$\cos x_1 \cos x_2 = \tfrac{1}{2}(1+B)\cos(\gamma/2).$$

Note that the number of free parameters in the boundary reflection matrices of both the vertex (the sine-Gordon model) and SOS class I, II representations are the same: two for the non-diagonal and one for the diagonal [5, 6]. This strongly suggests that a well-defined transformation between the two models can exist even with a boundary.

4. The hard hexagon model

The paticle spectrum of the HHM consists of a triplet of fundamental kink states K_{01}, K_{10} and K_{00} [9]. The bulk S-matrix is given [10, 11] by

$$S_{dc}^{ab}(\theta) = \mathcal{U}(\theta)\left(\frac{\rho_a\rho_c}{\rho_d\rho_b}\right)^{-\theta/2\pi\mathrm{i}} W_{dc}^{ab}(u) \tag{4.1}$$

with Boltzmann weights

$$W_{dc}^{ab}(u) = \left(\frac{\rho_a\rho_c}{\rho_d\rho_b}\right)^{1/2}\frac{\sin u}{\sin(\mu-u)}\delta_{bd} + \delta_{ac} \tag{4.2}$$

where

$$\rho_0 = 2\cos\mu \qquad \rho_1 = 1$$

$$\mu = \frac{\pi}{5} \qquad u = \frac{9\mathrm{i}}{5}\theta$$

$$\mathcal{U}(\theta) = \mathcal{U}_0(\theta)\,\mathcal{U}_1(u)$$

$$\mathcal{U}_0(\theta) = -\frac{\sinh\theta - \mathrm{i}\sin(\pi/9)}{\sinh\theta + \mathrm{i}\sin(\pi/9)}\;\frac{\sinh\theta + \mathrm{i}\sin(2\pi/9)}{\sinh\theta - \mathrm{i}\sin(2\pi/9)}$$

$$\mathcal{U}_1(u) = \frac{\sin(\mu-u)}{\sin(\mu+u)}\;\frac{\sin(2\mu+u)}{\sin(2\mu-u)}.$$

By mapping $a = 2, 3$ $(1, 4)$ of the RSOS(5) to $a = 0$ (1) of the HHM, one can reproduce the bulk S-matrix of the HHM from that of the RSOS(5). This means the RSOS(5) is homeomorphic to the HHM with differences in the overall factor $\mathcal{U}(u)$ and the relations between the spectral parameter u and the rapidity θ.

These mean that the two BYBEs can be mapped to each other and the solutions of the HHM can be obtained by that of the RSOS. Writing the reflection amplitude of the HHM as

$$R_{bc}^{a}(\theta) = \mathcal{R}(\theta)\left(\frac{\rho_b\rho_c}{\rho_a\rho_a}\right)^{-\theta/2\pi\mathrm{i}}\left[\delta_{b\neq c}X_{bc}^{a}(u) + \delta_{bc}\left\{\delta_{b,a+1}U(u) + \delta_{b,a}V(u) + \delta_{b,a-1}D(u)\right\}\right] \tag{4.3}$$

the non-diagonal solution is

$$
\begin{aligned}
U(u) &= A\sin(2u+3\mu) + \frac{B}{\sin 2u} + \frac{D}{\sin 3\mu}\left\{\frac{\sin(2u+3\mu)}{\sin 2u}+1\right\} \\
V(u) &= A\sin(2u-3\mu) + \frac{B}{\sin 2u} - \frac{D}{\sin 3\mu}\left\{\frac{\sin(2u-3\mu)}{\sin 2u}+1\right\} \\
D(u) &= A\sin(2u+\mu) + \frac{B}{\sin 2u} + \frac{D}{\sin \mu}\left\{\frac{\sin(2u+\mu)}{\sin 2u}+1\right\} \\
X_{01}^0 X_{10}^0 &= A^2\left(\sin^2\mu - \sin^2 3\mu\right) - 2AB\,(\cos\mu - \cos 3\mu) \\
&\quad + D^2\left(\frac{1}{\sin^2(\mu/2)} - \frac{1}{\sin^2(3\mu/2)}\right)
\end{aligned}
\tag{4.4}
$$

which can easily be read from equations (2.19),(2.20).

The unitarity and crossing symmetry conditions now reduce to

$$\mathcal{R}(\theta)\mathcal{R}(-\theta)D(u)D(-u) = 1 \tag{4.5}$$

$$\mathcal{R}(\pi\mathrm{i}/2-\theta) = \mathcal{U}_0(2\theta)\frac{\sin(2\mu+2u)}{\sin(2\mu-2u)}\mathcal{R}(\pi\mathrm{i}/2+\theta). \tag{4.6}$$

It is remarkable that for the HHM the unitarity condition cannot further reduce the arbitrary coefficients A, B and D. Of these the B and D terms are 'decorated' solutions which can be constructed from a fundamental solution R^a_{bc} by

$$\mathbf{R}^e_{fd}(u)_{bf;cd} = \sum_a S^{fe}_{ba}(u-u_1)S^{ae}_{cd}(u+u_1)R^a_{bc}(u). \tag{4.7}$$

It is easy to check this satisfies the BYBE if $S^{fe}_{ba}(u)$ is the solution of the bulk YBE with arbitrary u_1. Using the trivial solution $R^a_{bc} \propto \delta_{bc}$, one can check that B and D terms can be obtained in this way. We will therefore set $B = D = 0,\ A = 1$ from now on.

The overall factor $\mathcal{R}(\theta)$ is can be determined from equations (4.5), (4.6). Let

$$\mathcal{R}(\theta) = \mathcal{R}_0(\theta)\mathcal{R}_1(u) \tag{4.8}$$

such that

$$
\begin{aligned}
&\mathcal{R}_0(\theta)\mathcal{R}_0(-\theta) = 1 \qquad \mathcal{R}_0(\pi i/2-\theta) = \mathcal{U}_0(2\theta)\mathcal{R}_0(\pi\mathrm{i}/2+\theta) \\
&\mathcal{R}_1(u)\mathcal{R}_1(-u)\left(-\sin^2 2u + \sin^2\mu\right) = 1 \\
&\mathcal{R}_1(\mu/2-\pi-u) = \frac{\sin(2\mu+2u)}{\sin(2\mu-2u)}\mathcal{R}_1(\mu/2-\pi+u)
\end{aligned}
\tag{4.9}
$$

then the minimal solutions are

$$
\begin{aligned}
\mathcal{R}_0(\theta) &= \frac{\sinh(\theta/2+\pi\mathrm{i}/4)}{\sinh(\theta/2-\pi\mathrm{i}/4)}\frac{\sinh(\theta/2-\pi\mathrm{i}/36)}{\sinh(\theta/2+\pi\mathrm{i}/36)}\frac{\sinh(\theta/2+5\pi\mathrm{i}/18)}{\sinh(\theta/2-5\pi\mathrm{i}/18)} \\
&\quad\times\frac{\sinh(\theta/2+\pi\mathrm{i}/18)}{\sinh(\theta/2-\pi\mathrm{i}/18)}\frac{\sinh(\theta/2+7\pi\mathrm{i}/36)}{\sinh(\theta/2-7\pi\mathrm{i}/36)} \\
\mathcal{R}_1(u) &= \frac{1}{\sin(\mu+2u)}.
\end{aligned}
$$

By generalizing the mapping of the RSOS(5) to the HHM, we can construct some generalized HHMs whose particle spectrum consists of kinks K_{ab} where $|a-b| = 1$ with

$a, b = 0, \ldots, n-1$ and K_{00}. Denoting these as HHM(n), the bulk S-matrix of the HHM(n) can be obtained from that of the RSOS(2n + 1) by folding the heights as

$$a, \quad 2n+1-a \quad \rightarrow \quad n-a \qquad 1 \leqslant a \leqslant n. \tag{4.10}$$

The above one is well-defined without ambiguity, due to the symmetry of the S-matrix. The integrability conditions such as the YBE and BYBE are transformed accordingly, maintaing the structure. These mean that we can write the reflection amplitude of the HHM(n) as

$$R^a_{bc}(\theta) = \mathcal{R}(\theta) \left(\frac{\rho_b \rho_c}{\rho_a \rho_a}\right)^{-\theta/2\pi \mathrm{i}} \left[\delta_{b\neq c} X^a_{bc}(u) + \delta_{bc} \left\{\delta_{b,a+1} U_a(u) + \delta_{b,a} V(u) + \delta_{b,a-1} D_a(u)\right\}\right] \tag{4.11}$$

where ρ_a denotes the q-number

$$\rho_a = \frac{\sin \bar{a}\lambda}{\sin \lambda} \qquad \bar{a} = n-a \qquad \lambda = \frac{\pi}{2n+1}.$$

Then the solution of the BYBEs for the HHM(n) are

$$\begin{aligned}
U_a(u) &= (-1)^{\bar{a}+1} A \sin(2u - \bar{a}\lambda) + \frac{B}{\sin 2u} - \frac{D}{\sin \bar{a}\lambda} \left\{\frac{\sin(2u - \bar{a}\lambda)}{\sin 2u} - (-1)^{\bar{a}}\right\} \\
D_a(u) &= (-1)^{\bar{a}+1} A \sin(2u + \bar{a}\lambda) + \frac{B}{\sin 2u} + \frac{D}{\sin \bar{a}\lambda} \left\{\frac{\sin(2u + \bar{a}\lambda)}{\sin 2u} - (-1)^{\bar{a}}\right\} \\
V(u) &= (-1)^{n+1} A \sin(2u + n\lambda) + \frac{B}{\sin 2u} + \frac{D}{\sin n\lambda} \left\{\frac{\sin(2u + n\lambda)}{\sin 2u} - (-1)^{n}\right\} \\
X^a_{a-1,a+1} X^a_{a+1,a-1} &= A^2 \left\{\sin^2 \lambda - \sin^2 \bar{a}\lambda\right\} - 2AB \left\{\cos \lambda - (-1)^{\bar{a}+1} \cos \bar{a}\lambda\right\} \\
&\quad + 2D^2 \left\{\frac{1}{\sin^2 \lambda} - \frac{1}{\sin^2 \bar{a}\lambda}\right\} + 2D^2 \left\{\frac{\cos \lambda}{\sin^2 \lambda} - (-1)^{\bar{a}+1} \frac{\cos \bar{a}\lambda}{\sin^2 \bar{a}\lambda}\right\}
\end{aligned} \tag{4.12}$$

which can be read from equations (2.19), (2.20) with a odd.

5. Conclusion

In this paper we derived the most general non-diagonal reflection matrices of the RSOS/SOS models and the hard hexagon model using the boundary Yang–Baxter equations. We find a new one-parameter family of the reflection matrices for the RSOS model which generalizes the previous result in [6] where there is no free parameter. This free parameter can be used to control the flow between the fixed and free boundary conditions. Since the bulk RSOS theory describes the perturbed conformal theories by the least relevent operator, the boundary conditions can show how the conformal boundary conditions can change under renormalization group flows.

The vertex–face correspondence between the sine-Gordon and SOS theories in the presence of a boundary remains an open problem: our three classes of SOS reflection matrices may be useful for this purpose.

For the hard-hexagon model which can be mapped to the RSOS(5) model by folding four RSOS heights into two, the solutions can be obtained similarly with a main difference in the boundary unitarity conditions. Due to this, the reflection matrices can have two free parameters. We show that these extra terms can be identified with the ‘decorated’ solutions. This means the general solutions of the BYBE are linear combinations of fundamental solutions and their decorated ones.

Considering that the HHM is related to the perturbed conformal theory by the most relevent operator, it will be interesting to consider how the two different perturbations can make difference in the boundary interactions.

Acknowledgments

CA is supported in part by KOSEF 961-0201-006-2, 95-0701-04-01-3, BSRI 97-2427 and by a grant from KOSEF through CTP/SNU. CKY is supported by a postdoctoral grant through the Korea Research Foundation.

References

[1] Cherednik I V 1984 *Teor. Mat. Fiz.* **61** 35
[2] Sklyanin E K 1988 *J. Phys. A: Math. Gen.* **21** 2375
[3] Mezincescu L and Nepomechie R I 1991 *Int. J. Mod. Phys.* A **6** 5231
[4] Goshal S and Zamolodchikov A B 1994 *Int. J. Mod. Phys.* A **9** 3841
[5] Behrend R E, Pearce P A and O'Brien D L 1996 *J. Stat. Phys.* **84** 1
[6] Ahn C and Koo W M 1996 *Nucl. Phys.* B **468** [FS] 461
[7] Miwa T and Weston R 1997 *Nucl. Phys.* B **486** [PM] 517
[8] Ahn C and Koo W M 1996 *J. Phys. A: Math. Gen.* **29** 5845
[9] Lässig M, Mussardo G and Cardy J L 1991 *Nucl. Phys.* B **348** 591
[10] Koubek A 1994 *Int. J. Mod. Phys.* A **9** 1909
[11] Ellem R M and Bazhanov V V 1997 Thermodynamic Bethe ansatz for the subleading magnetic perturbation of the tricritical Ising model *Preprint* hep-th/9703026