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Boundary action of $N=2$ super-Liouville theory

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We derive a boundary action of $N=2$ super-Liouville theory which preserves both $N=2$ supersymmetry and conformal symmetry by imposing explicitly $T=\overline{T}$ and $G=\overline{G}$. The resulting boundary action shows a new duality symmetry.

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I. INTRODUCTION

Two-dimensional Liouville field theory (LFT) has been studied actively for its relevance to noncritical string theories and two-dimensional quantum gravity [1,2]. This theory has been extended to the supersymmetric Liouville field theories (SLFTs) which can describe the noncritical superstring theories. In particular, the $N=2$ SLFT has been studied actively because the world sheet supersymmetry can generate space-time supersymmetry. In addition to applications to string theories, these models provide theoretically challenging problems. The Liouville theory and its supersymmetric generalizations are irrational conformal field theories (CFTs) which have a continuously infinite number of primary fields. Because of this property, most CFT formalisms developed for rational CFTs do not apply to this class of model. An interesting problem is to extend the conventional CFT formalism to irrational CFTs. There has been a lot of progress in this field. Various methods have been proposed to derive structure constants and reflection amplitudes, which are basic building blocks to complete the conformal bootstrap [3–5]. These have been extended to the $N=1$ SLFT in [6,7].

A more challenging problem is to extend these formalisms to the CFTs defined in the two-dimensional space-time geometry with a boundary condition (BC) which preserves the conformal symmetry. Cardy showed that the conformally invariant BCs can be associated with the primary fields in terms of modular S-matrix elements for the case of rational CFTs [8]. It has been an issue whether the Cardy formalism can be extended to the irrational CFTs. There are active efforts to understand the conformally invariant boundary states in the context of string theories related to D-branes [9,10].

Important progress in this direction was made in [11] where the functional relation method developed in [4] was used in the boundary LFT. With a boundary action which preserves conformal symmetry, a one-point function of a bulk operator in the presence of the boundary interaction and two-point correlation functions of boundary operators have been computed using the functional relation method [11]. Here the conformal BC is denoted by a continuous parameter appearing in the boundary action. A similar treatment of the LFT defined in the classical Lobachevskiy plane—namely, the pseudosphere—has been made in [12]. For the $N=1$ SLFT, the one-point functions and the boundary two-point functions have been obtained in [13,14] based on the conjectured boundary action. It is desirable to show that indeed this action preserves both supersymmetry and conformal symmetry.

In this paper we derive the boundary actions of the $N=1,2$ SLFTs by imposing the symmetries. This approach to obtain the boundary actions has been made before. In [15], based on a superfield formulation, the $N=2$ supersymmetric boundary action has been derived for a general $N=2$ supersymmetric quantum field theory. For integrable quantum field theories with infinite conserved charges, the situation becomes much more complicated. As shown in a pioneering work [16], the boundary action which preserves the integrability can be fixed by imposing a first few conservation laws. For the supersymmetric integrable models, the two conditions—the supersymmetry and integrability—have been successfully imposed to get appropriate boundary actions [17–19]. We continue this approach to the $N=1,2$ SLFTs and impose the boundary superconformal invariance conditions to derive the boundary actions. We will show that even at the classical level, the boundary actions are determined uniquely.

This paper is organized as follows. In Sec. II we review a superfield formulation of the $N=1$ SLFT boundary action proposed previously. Then, we show that this action satisfies the superconformal invariance. Our main result—the superconformally invariant boundary action of the $N=2$ SLFT—is derived in Sec. III. After repeating the superfield formulation, we derive the boundary action by imposing $N=2$ superconformal symmetry. We conclude in Sec. IV with a few discussions and provide technical details in the Appendixes.

II. BOUNDARY $N=1$ SUPER-LIOUVILLE THEORY

In this section, we review a superfield formulation of the boundary action of the $N=1$ SLFT which preserves the boundary $N=1$ supersymmetry. Then, we will show that the same result can be obtained by imposing directly the $N=1$ superconformal symmetry.

A. Superfield formulation of the $N=1$ boundary action

The action of the $N=1$ SLFT is given by [20]

$$S=\int d^2z d^2\theta\left(\frac{1}{2\pi}\overline{D}\Phi D\Phi+i\mu e^{b\Phi}\right), \tag{2.1}$$

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where Φ is a real scalar superfield:

$$\Phi=\phi+i\theta\psi-i\overline{\theta\psi}+i\theta\overline{\theta}F. \tag{2.2}$$

(See Appendix A 1 for our conventions of the $N=1$ supersymmetry.) This theory contains a dimensionless Liouville coupling constant b and the cosmological constant μ. Note that we consider a trivial background and omit a linear dilaton coupling. We can express the action in terms of the component fields,

$$S=\int d^2z\left[\frac{1}{2\pi}(\partial\phi\overline{\partial}\phi+\psi\overline{\partial}\psi+\overline{\psi}\partial\overline{\psi})+i\mu b^2\psi\overline{\psi}e^{b\phi}+\frac{1}{2}\pi\mu^2b^2e^{2b\phi}\right], \tag{2.3}$$

by integrating over the θ and $\overline{\theta}$ coordinates in Eq. (2.1) and eliminating the auxiliary field F from its equation of motion.

To introduce the boundary action, we consider first a general $N=1$ supersymmetric theory on the lower half-plane: $-\infty<x=\mathrm{Re}\,z<\infty$, $-\infty<y=\mathrm{Im}\,z\leqslant 0$. Following [10], we can write the action as follows:

$$S=\int_{-\infty}^{\infty}dx\int_{-\infty}^{0}dy\int d^2\theta\mathcal{L}, \tag{2.4}$$

where $\mathcal{L}$ is the Lagrangian density in superspace. The supersymmetry variation of the action is

$$\delta S=\int_{-\infty}^{\infty}dx\int_{-\infty}^{0}dy\int d^2\theta(\zeta Q+\overline{\zeta Q})\mathcal{L}$$
$$=-\frac{i}{2}\int_{-\infty}^{\infty}dx(\zeta\mathcal{L}|_{\overline{\theta}}+\overline{\zeta}\mathcal{L}|_{\theta})|_{y=0}. \tag{2.5}$$

To cancel the surface term (2.5), we add a boundary action

$$S_B=\frac{i}{2}\eta\int_{-\infty}^{\infty}dx\mathcal{L}|_{\theta=\overline{\theta}=0},\quad \eta=\pm 1, \tag{2.6}$$

which is defined at $y=0$. When $\zeta=\eta\overline{\zeta}$, the supersymmetry variation of the total action vanishes: $\delta S+\delta S_B=0$. Only one supercharge $Q+\eta\overline{Q}$ is preserved. Conservation of this charge imposes the boundary condition on the supercurrent: $G+\eta\overline{G}=0$ at $y=0$. The superderivatives in the tangential and normal directions are given by $D_t=D+\eta\overline{D}$ and $D_n=D-\eta\overline{D}$, respectively. Their conjugate coordinates are $\theta_t=(\theta+\eta\overline{\theta})/2$ and $\theta_n=(\theta-\eta\overline{\theta})/2$.

For the total variation of $S+S_B$ to vanish, two types of boundary conditions can be imposed.

(i) Dirichlet boundary conditions $D_t\Phi|_{y=\theta_n=0}=0$: For the $N=1$ SLFT, this corresponds to

$$\psi-\eta\overline{\psi}|_{y=0}=0,\ \ \partial_x\phi|_{y=0}=0. \tag{2.7}$$

These conditions can be identified with the supersymmetric version of the ZZ brane [12–14]

(ii) Neumann boundary conditions $D_n\Phi|_{y=\theta_n=0}=0$: For the $N=1$ SLFT, these give

$$\psi+\eta\overline{\psi}|_{y=0}=0,\ \ \partial_y\phi-2\eta\pi\mu b e^{b\phi}|_{y=0}=0. \tag{2.8}$$

These boundary conditions correspond to the supersymmetric version of the FZZT brane [11,21].

In [10], it is shown that one can add an additional term to the boundary action

$$S_B'=-\frac{1}{2}\int_{-\infty}^{\infty}dx\int d\theta_t\left(\Gamma D_t\Gamma+\frac{4}{b}i\mu_B\Gamma e^{b\Phi/2}\right), \tag{2.9}$$

with a fermionic boundary superfield $\Gamma=a+i\theta_t h$. In fact, this boundary action is equivalent to that considered previously in [13,14]. We will show in the next subsection that this action indeed preserves the boundary superconformal symmetry.

B. Boundary superconformal symmetry

To derive a boundary action which preserves both $N=1$ supersymmetry and conformal symmetry, we start with a general form of boundary action

$$S_B=\int_{-\infty}^{\infty}dx\left[-\frac{i}{4\pi}\overline{\psi}\psi+\frac{1}{2}a\partial_x a-f(\phi)a(\psi+\overline{\psi})+B(\phi)\right], \tag{2.10}$$

where a is a real fermionic boundary degree of freedom which anticommutes with ψ and $\overline{\psi}$. The boundary action (2.10) was first proposed in the boundary $N=1$ supersymmetric sine-Gordon model [18]. $f(\phi)$ and $B(\phi)$ are functions of the scalar field ϕ to be determined by the boundary conditions which preserve $N=1$ supersymmetry. The fermionic boundary degree of freedom a was first introduced in the Ising model in a boundary magnetic field [16] and in the $N=1$ SLFT with appropriate kinetic term [14].

The boundary $N=1$ superconformal symmetry imposes the following constraints on the stress tensor and supercurrent:

$$T=\overline{T},\ \ G=\overline{G}\ \ \text{at}\ \ y=0. \tag{2.11}$$

Here we choose $\eta=-1$ and preserve only one supercharge $Q-\overline{Q}$. Hence, it is called sometimes as $N=1/2$ supersymmetry.

The stress tensor T and the supercurrent G are given by

$$T=-\frac{1}{2}[(\partial\phi)^2+\psi\partial\psi]+\frac{1}{2}\hat{Q}\partial^2\phi,\ \ G=i(\psi\partial\phi-\hat{Q}\partial\psi), \tag{2.12}$$

where $\hat{Q}$ is the background charge. By using the equations of motion, one can easily show that the conservation laws $\overline{\partial}T=\partial\overline{T}=\overline{\partial}G=\partial\overline{G}=0$ are satisfied at the classical level with $\hat{Q}=1/b$.

Using the bulk equations of motion,

$$\partial\bar{\partial}\phi=\pi\mu b^3(i\psi\bar{\psi}+\pi\mu e^{b\phi})e^{b\phi},$$

$$\bar{\partial}\psi=-\pi i\mu b^2\bar{\psi}e^{b\phi},\quad \partial\bar{\psi}=\pi i\mu b^2\psi e^{b\phi}, \tag{2.13}$$

and the boundary equations of motion,

$$\partial_y\phi=4\pi\frac{\partial f}{\partial\phi}a(\psi+\bar{\psi})-4\pi\frac{\partial B}{\partial\phi},$$

$$\psi-\bar{\psi}=-4\pi i f a,\quad \partial_x a=f(\psi+\bar{\psi}), \tag{2.14}$$

we obtain

$$G-\bar{G}=2\pi\left(f-\frac{2}{b}\frac{\partial f}{\partial\phi}\right)\partial_x\phi a$$
$$+\pi\left(-\frac{2}{f}\frac{\partial B}{\partial\phi}-\frac{4}{b}f+\mu b e^{b\phi}\frac{1}{f}\right)\partial_x a. \tag{2.15}$$

Here we eliminated $\psi,\bar{\psi}$ assuming f is not zero.

The condition $G-\bar{G}=0$ can be satisfied by the following f and B:

$$f=\mu_B e^{b\phi/2},\quad B=\left(-\frac{2}{b^2}\mu_B^2+\frac{1}{2}\mu\right)e^{b\phi}, \tag{2.16}$$

where μ_B is the boundary cosmological constant. One can show similarly that $T-\bar{T}=0$ can be also satisfied. One can easily check that the boundary action (2.9) with (2.6) in terms of the superfields is indeed the same as Eqs. (2.10) with (2.16). Therefore, this action preserves not only boundary $N=1$ supersymmetry but also conformal symmetry.

So far, we have considered only the classical equations of motion. Even at this level, the boundary action has been determined uniquely. We can consider quantum corrections in a similar approach. For this, we interpret $e^{b\phi}$ in Eqs. (2.13) as the normal-ordered exponential $:e^{b\phi}:$. The fields in the stress tensor and the supercurrent in Eq. (2.12) should be also normal ordered. With this change, we obtain

$$\bar{\partial}T=\pi\mu b^2(1+b^2-\hat{Q}b)[\pi\mu\partial(:e^{b\phi}:)^2$$
$$+i\psi\bar{\psi}\partial:e^{b\phi}:-i\bar{\psi}\partial\psi:e^{b\phi}:]. \tag{2.17}$$

The conservation law $\bar{\partial}T=0$ (and others) can be satisfied when the background charge is renormalized to $\hat{Q}=1/b+b$. We will show in Appendix B that the boundary superconformal symmetry $T-\bar{T}=0$ and $G-\bar{G}=0$ is also preserved at the quantum level with this $\hat{Q}$.

III. BOUNDARY $N=2$ SUPER-LIOUVILLE THEORY

In this section, we use previous method to derive the superconformal boundary action of the $N=2$ SLFT.

A. Superfield formulation

The action of the $N=2$ SLFT is given by

$$S=\int d^2z\left[\frac{1}{\pi}\int d^4\theta\Phi^+\Phi^-+\left(i\mu\int d^2\theta^+e^{b\Phi^+}+\text{c.c.}\right)\right], \tag{3.1}$$

where $\Phi^\pm$ are the chiral superfields which satisfy

$$D_\mp\Phi^\pm=\bar{D}_\mp\Phi^\pm=0. \tag{3.2}$$

Therefore, $\Phi^\pm$ can be expanded as

$$\Phi^\pm=\phi^\pm(y^\pm,\bar{y}^\pm)+i\theta^\pm\psi^\mp(y^\pm,\bar{y}^\pm)-i\bar{\theta}^\pm\bar{\psi}^\mp(y^\pm,\bar{y}^\pm)$$
$$+i\theta^\pm\bar{\theta}^\pm F^\pm(y^\pm,\bar{y}^\pm), \tag{3.3}$$

where $y^\pm=z+\frac{1}{2}\theta^\pm\theta^\mp$ and $\bar{y}^\pm=\bar{z}+\frac{1}{2}\bar{\theta}^\pm\bar{\theta}^\mp$. (See Appendix A 2 for conventions.) The action can be written in terms of the component fields as

$$S=\int d^2z\left[\frac{1}{2\pi}(\partial\phi^-\bar{\partial}\phi^++\partial\phi^+\bar{\partial}\phi^-+\psi^-\bar{\partial}\psi^+\right.$$
$$+\psi^+\bar{\partial}\psi^-+\bar{\psi}^-\partial\bar{\psi}^++\bar{\psi}^+\partial\bar{\psi}^-)+i\mu b^2\psi^-\bar{\psi}^-e^{b\phi^+}$$
$$\left.+i\mu b^2\psi^+\bar{\psi}^+e^{b\phi^-}+\pi\mu^2b^2e^{b(\phi^++\phi^-)}\right]. \tag{3.4}$$

Now we consider boundary conditions in the $N=2$ SLFT on the lower half-plane. The action can be written as

$$S=\int_{-\infty}^{\infty}dx\int_{-\infty}^{0}dy\left[\int d^4\theta K(\Phi^+,\Phi^-)\right.$$
$$\left.+\int d^2\theta^+W^+(\Phi^+)-\int d^2\theta^-W^-(\Phi^-)\right]$$
$$=S_K+S_W, \tag{3.5}$$

where $K(\Phi^+,\Phi^-)$ is a Kähler potential and $W^\pm(\Phi^\pm)$ are superpotentials. Consider first the case where only the Kähler potential term exists [22]. The supersymmetric variation of S_K is

$$\delta S_K=\int_{-\infty}^{\infty}dx\int_{-\infty}^{0}dy\int d^4\theta(\zeta^+Q_++\bar{\zeta}^+\bar{Q}_++\zeta^-Q_-$$
$$+\bar{\zeta}^-\bar{Q}_-)K(\Phi^+,\Phi^-)$$
$$=\frac{i}{4}\int_{-\infty}^{\infty}dx(\zeta^+K|_{\theta^+\bar{\theta}^+\bar{\theta}^-}+\bar{\zeta}^+K|_{\theta^+\bar{\theta}^+\theta^-}$$
$$+\zeta^-K|_{\bar{\theta}^+\theta^-\bar{\theta}^-}+\bar{\zeta}^-K|_{\theta^+\theta^-\bar{\theta}^-})|_{y=0}. \tag{3.6}$$

We can cancel Eq. (3.6) by adding two types of boundary actions

$$S_{BK}=\frac{i}{4}\int_{-\infty}^{\infty}dx(e^{i\beta}K|_{\bar{\theta}^+\theta^+}+e^{-i\beta}K|_{\bar{\theta}^-\theta^-}) \tag{3.7}$$

and

$$S_{BK}=\frac{i}{4}\int_{-\infty}^{\infty}dx(e^{i\beta}K|_{\theta^+\bar{\theta}^-}+e^{-i\beta}K|_{\theta^-\bar{\theta}^+}), \quad (3.8)$$

where $e^{i\beta}$ is an arbitrary phase. In the first case, the supersymmetry variation of S_K+S_{BK} vanishes when $\overline{\zeta}^{\pm}=e^{\pm i\beta}\zeta^{\mp}$. The conserved supercharges are $Q_++e^{-i\beta}\overline{Q}_-$ and $Q_-+e^{i\beta}\overline{Q}_+$. This leads to a condition on the supercurrents: $G^{\pm}+e^{\mp i\beta}\overline{G^{\mp}}=0$ at $y=0$. This case is called A-type boundary conditions [23]. The second case is $\overline{\zeta}^{\pm}=e^{\mp i\beta}\zeta^{\pm}$ where conserved supercharges are $Q_++e^{-i\beta}\overline{Q}_+$ and $Q_-+e^{i\beta}\overline{Q}_-$. Associated boundary conditions on the supercurrents will be called B-type boundary condition: $G^{\pm}+e^{\mp i\beta}\overline{G}^{\pm}=0$ at $y=0$. In this paper, we will consider $e^{i\beta}=-1$ for simplicity.

With nonvanishing superpotential $W^{\pm}$, the supersymmetric variation becomes

$$\delta S_W=\frac{1}{2}\int_{-\infty}^{\infty}dx\Bigg[(\overline{\zeta}^-\psi^--\zeta^-\overline{\psi}^-)\frac{\partial W^+}{\partial\phi^+} + (\zeta^+\overline{\psi}^+-\overline{\zeta}^+\psi^+)\frac{\partial W^-}{\partial\phi^-}\Bigg]. \quad (3.9)$$

We classify the boundary conditions into two classes following [15,22].

1. A-type boundary condition

We set $\overline{\zeta}^{\pm}=-\zeta^{\mp}$ in Eq. (3.9) and assume that the fermions satisfy the condition

$$\psi^{\pm}-\overline{\psi}^{\mp}|_{y=0}=0. \quad (3.10)$$

The boundary conditions for the bosons are given by

$$\partial_x(\phi^+-\phi^-)=0,\ \ \partial_y(\phi^++\phi^-)=0. \quad (3.11)$$

If the superpotentials $W^{\pm}$ satisfy

$$\frac{\partial W^+}{\partial\phi^+}-\frac{\partial W^-}{\partial\phi^-}\bigg|_{y=0}=0, \quad (3.12)$$

$\delta S_W=0$ can be achieved.

2. B-type boundary condition

If $\overline{\zeta}^{\pm}=-\zeta^{\pm}$, Eq. (3.9) becomes

$$\delta S_W=\frac{1}{2}\int_{-\infty}^{\infty}dx\Bigg[-\zeta^-(\psi^-+\overline{\psi}^-)\frac{\partial W^+}{\partial\phi^+} + \zeta^+(\psi^++\overline{\psi}^+)\frac{\partial W^-}{\partial\phi^-}\Bigg]. \quad (3.13)$$

This vanishes for two types of boundary conditions.

(i) Dirichlet boundary conditions

$$\psi^{\pm}+\overline{\psi}^{\pm}|_{y=0}=0,\ \ \partial_x\phi^{\pm}|_{y=0}=0. \quad (3.14)$$

(ii) Neumann boundary conditions

$$\psi^{\pm}-\overline{\psi}^{\pm}|_{y=0}=0,\ \ \partial_y\phi^{\pm}|_{y=0}=0. \quad (3.15)$$

While no additional condition is needed for (i), the additional conditions

$$\frac{\partial W^{\pm}}{\partial\phi^{\pm}}\bigg|_{y=0}=0 \quad (3.16)$$

are necessary for case (ii). To avoid this unphysical situation, one must add an additional boundary term

$$S_{BW}=\frac{i}{2}\eta\int_{-\infty}^{\infty}dx(W^+-W^-)\bigg|_{\theta^{\pm}=\overline{\theta}^{\pm}=0}. \quad (3.17)$$

The variation of this term cancels δS_W in Eq. (3.13) if $\zeta^-+\eta\zeta^+=0$ is satisfied. This leads to the boundary conditions for $\phi^{\pm}$:

$$\partial_y\phi^{\pm}\mp 2\pi i\eta\frac{\partial W^{\mp}}{\partial\phi^{\mp}}\bigg|_{y=0}=0. \quad (3.18)$$

Therefore, only $N=1$ supersymmetry is preserved.

B. Boundary action of $N=2$ super-Liouville theory

Here we construct the boundary action with a B-type boundary condition which preserves $N=2$ superconformal invariance. We start with

$$S_B=\int_{-\infty}^{\infty}dx\Bigg[-\frac{i}{4\pi}(\overline{\psi}^+\psi^-+\overline{\psi}^-\psi^+)+\frac{1}{2}a^-\partial_x a^+ -\frac{1}{2}(f^+(\phi^+)a^++\tilde{f}^+(\phi^+)a^-)(\psi^-+\overline{\psi}^-) -\frac{1}{2}(f^-(\phi^-)a^-+\tilde{f}^-(\phi^-)a^+)(\psi^++\overline{\psi}^+) +B(\phi^+,\phi^-)\Bigg], \quad (3.19)$$

where $a^{\pm}$ are complex fermionic boundary degrees of freedom, which anticommute with $\psi^{\pm}$ and $\overline{\psi}^{\pm}$. The boundary action of the form (3.19) was first proposed in the context of the $N=2$ supersymmetric sine-Gordon model [19]. $f^{\pm}(\phi^{\pm})$, $\tilde{f}^{\pm}(\phi^{\pm})$, and $B(\phi^+,\phi^-)$ are functions of $\phi^{\pm}$ to be determined by the boundary conditions.

The stress tensor T, the supercurrent $G^{\pm}$, and the $U(1)$ current J are given by

$$T=-\partial\phi^-\partial\phi^+-\frac{1}{2}(\psi^-\partial\psi^++\psi^+\partial\psi^-) +\frac{1}{2}\hat{Q}(\partial^2\phi^++\partial^2\phi^-), \quad (3.20)$$

$$G^{\pm}=\sqrt{2}i(\psi^{\pm}\partial\phi^{\pm}-\hat{Q}\partial\psi^{\pm}), \tag{3.21}$$

$$J=-\psi^{-}\psi^{+}+\hat{Q}(\partial\phi^{+}-\partial\phi^{-}), \tag{3.22}$$

where $\hat{Q}$ is the background charge.

One can show that the conservation laws $\bar{\partial}T=\partial\bar{T}=\bar{\partial}G^{\pm}=\partial\bar{G}^{\pm}=\bar{\partial}J=\partial\bar{J}=0$ are satisfied at the classical level when $\hat{Q}=1/b$ in the same way as the $N=1$ case. One major difference for the $N=2$ SLFT is that $\hat{Q}$ has no quantum correction. The above conservation laws hold at the quantum level with $\hat{Q}=1/b$ due to $:e^{b\phi^{+}}::e^{b\phi^{-}}:=:e^{b\phi^{-}}::e^{b\phi^{+}}:$. This means that a classical level computation is sufficient for our consideration. Also, without the correction, dual symmetry $b\to 1/b$ disappears. The lack of dual symmetry makes it much harder to solve even bulk $N=2$ SLFT [24].

To preserve $N=2$ superconformal symmetry, we impose the following boundary conditions on the conserved currents:

$$T=\bar{T},\quad G^{\pm}=\bar{G}^{\pm},\quad J=\bar{J}\quad \text{at}\quad y=0. \tag{3.23}$$

Substituting the bulk equations of motion,

$$\partial\bar{\partial}\phi^{\pm}=\pi\mu b^{3}(i\psi^{\pm}\bar{\psi}^{\pm}+\pi\mu e^{b\phi^{\pm}})e^{b\phi^{\mp}},$$

$$\bar{\partial}\psi^{\pm}=-\pi i\mu b^{2}\bar{\psi}^{\mp}e^{b\phi^{\pm}},$$

$$\partial\bar{\psi}^{\pm}=\pi i\mu b^{2}\psi^{\mp}e^{b\phi^{\pm}}, \tag{3.24}$$

and the boundary equations of motion,

$$\partial_{y}\phi^{\pm}=2\pi\left(\frac{\partial f^{\mp}}{\partial\phi^{\mp}}a^{\mp}+\frac{\partial\tilde{f}^{\mp}}{\partial\phi^{\mp}}a^{\pm}\right)(\psi^{\pm}+\bar{\psi}^{\pm})-4\pi\frac{\partial B}{\partial\phi^{\mp}},$$

$$\psi^{\pm}-\bar{\psi}^{\pm}=-2\pi i(f^{\pm}a^{\pm}+\tilde{f}^{\pm}a^{\mp}),$$

$$\partial_{x}a^{\pm}=f^{\mp}(\psi^{\pm}+\bar{\psi}^{\pm})+\tilde{f}^{\pm}(\psi^{\mp}+\bar{\psi}^{\mp}), \tag{3.25}$$

into $G^{\pm}-\bar{G}^{\pm}$ and eliminating $\psi^{\pm}$ and $\bar{\psi}^{\pm}$, we obtain

$$G^{\pm}-\bar{G}^{\pm}=\pi\left(f^{\pm}-\frac{2}{b}\frac{\partial f^{\pm}}{\partial\phi^{\pm}}\right)\partial_{x}\phi^{\pm}a^{\pm}+\pi\left(\tilde{f}^{\pm}-\frac{2}{b}\frac{\partial\tilde{f}^{\pm}}{\partial\phi^{\pm}}\right)\partial_{x}\phi^{\pm}a^{\mp}+\pi\left(-\frac{2f^{\pm}}{f^{\pm}f^{\mp}-\tilde{f}^{\pm}\tilde{f}^{\mp}}\frac{\partial B}{\partial\phi^{\mp}}-\frac{2}{b}f^{\pm}\right.$$
$$\left.-\frac{\mu b\tilde{f}^{\mp}}{f^{\pm}f^{\mp}-\tilde{f}^{\pm}\tilde{f}^{\mp}}e^{b\phi^{\pm}}\right)\partial_{x}a^{\pm}+\pi\left(\frac{2\tilde{f}^{\pm}}{f^{\pm}f^{\mp}-\tilde{f}^{\pm}\tilde{f}^{\mp}}\frac{\partial B}{\partial\phi^{\mp}}-\frac{2}{b}\tilde{f}^{\pm}+\frac{\mu bf^{\mp}}{f^{\pm}f^{\mp}-\tilde{f}^{\pm}\tilde{f}^{\mp}}e^{b\phi^{\pm}}\right)\partial_{x}a^{\mp}. \tag{3.26}$$

The condition $G^{\pm}-\bar{G}^{\pm}=0$ determines $f^{\pm}$, $\tilde{f}^{\pm}$, and B as follows:

$$f^{\pm}=C^{\pm}e^{b\phi^{\pm}/2},\ \tilde{f}^{\pm}=\tilde{C}^{\pm}e^{b\phi^{\pm}/2}, \tag{3.27}$$

$$B=-\frac{2}{b^{2}}(C^{+}C^{-}+\tilde{C}^{+}\tilde{C}^{-})e^{b(\phi^{+}+\phi^{-})/2}, \tag{3.28}$$

where $C^{\pm}$ and $\tilde{C}^{\pm}$ are complex constants which obey $C^{\pm}\tilde{C}^{\pm}=\mu b^{2}/4$.

We next consider the stress tensor. Eliminating $a^{\pm}$ from Eq. (3.25) and using Eq. (3.27) and (3.28), we obtain

$$\partial_{y}\phi^{\pm}=i\frac{b}{2}(\psi^{\mp}-\bar{\psi}^{\mp})(\psi^{\pm}+\bar{\psi}^{\pm})$$
$$+\frac{4\pi}{b}(C^{+}C^{-}+\tilde{C}^{+}\tilde{C}^{-})e^{b(\phi^{+}+\phi^{-})/2}, \tag{3.29}$$

$$\partial_{x}\psi^{\pm}-\partial_{x}\bar{\psi}^{\pm}=\frac{b}{2}\partial_{x}\phi^{\pm}(\psi^{\pm}-\bar{\psi}^{\pm})$$
$$-2\pi i(C^{+}C^{-}+\tilde{C}^{+}\tilde{C}^{-})e^{b(\phi^{+}+\phi^{-})/2}$$
$$\times(\psi^{\pm}+\bar{\psi}^{\pm})-4\pi iC^{\pm}\tilde{C}^{\pm}e^{b\phi^{\pm}}(\psi^{\mp}+\bar{\psi}^{\mp}). \tag{3.30}$$

Substituting the above equations into $T-\bar{T}$ and $J-\bar{J}$, one can show that our solution satisfies both $T=\bar{T}$ and $J=\bar{J}$.

We have obtained the boundary action (3.19) with Eqs. (3.27) and (3.28). Moreover, we impose the invariance of $\mathcal{L}_{B}$ under complex conjugation. This invariance implies $C^{+}=(C^{-})^{*}$ and $C^{\pm}$ can be written as $C^{\pm}=\mu_{B}e^{\pm i\alpha}$, where α is a real parameter. This phase factor can be gauged away by redefining the fermionic zero modes $a^{\pm}\to e^{\mp i\alpha}a^{\pm}$. Therefore, the final form of the boundary action is

$$S_{B}=\int_{-\infty}^{\infty}dx\left[-\frac{i}{4\pi}(\bar{\psi}^{+}\psi^{-}+\bar{\psi}^{-}\psi^{+})+\frac{1}{2}a^{-}\partial_{x}a^{+}\right.$$
$$-\frac{1}{2}e^{b\phi^{+}/2}\left(\mu_{B}a^{+}+\frac{\mu b^{2}}{4\mu_{B}}a^{-}\right)(\psi^{-}+\bar{\psi}^{-})$$
$$-\frac{1}{2}e^{b\phi^{-}/2}\left(\mu_{B}a^{-}+\frac{\mu b^{2}}{4\mu_{B}}a^{+}\right)(\psi^{+}+\bar{\psi}^{+})$$
$$\left.-\frac{2}{b^{2}}\left(\mu_{B}^{2}+\frac{\mu^{2}b^{4}}{16\mu_{B}^{2}}\right)e^{b(\phi^{+}+\phi^{-})/2}\right]. \tag{3.31}$$

This is the main result of this paper. This action preserves two conserved supercharges $Q_{+}-\bar{Q}_{+}$ and $Q_{-}-\bar{Q}_{-}$.

We can rewrite this boundary action in terms of boundary superfields. Defining supertranslation operators $D_{t\pm}=D_{\pm}-\bar{D}_{\pm}$, which satisfy

$$\{D_{t+},D_{t-}\}=\partial_x,\quad D_{t+}^2=D_{t-}^2=0, \tag{3.32}$$

and their conjugate coordinates $\theta_t^{\pm}=(\theta^{\pm}-\bar{\theta}^{\pm})/2$, we introduce fermionic boundary chiral superfields $\Gamma^{\pm}$:

$$D_{t\mp}\Gamma^{\pm}=0. \tag{3.33}$$

The boundary superfields $\Gamma^{\pm}$ can be expanded as

$$\Gamma^{\pm}=a^{\pm}(x^{\pm})+i\theta_t^{\pm}h^{\pm}(x^{\pm}), \tag{3.34}$$

where $x^{\pm}=x+\frac{1}{2}\theta_t^{\pm}\theta_t^{\mp}$. In terms of these superfields, the boundary action can be written as

$$S_B=\int_{-\infty}^{\infty}dx\left\{-\frac{i}{4\pi}(\Phi^+\Phi^-|_{\theta^+\bar{\theta}^-}+\Phi^+\Phi^-|_{\theta^-\bar{\theta}^+})-\frac{1}{2}\mu(e^{b\Phi^+}+e^{b\Phi^-})\right|_{\theta^{\pm}=\bar{\theta}^{\pm}=0}$$
$$+\frac{1}{2}\int d\theta_t^+d\theta_t^-\Gamma^+\Gamma^- -\left[\frac{i}{b}\int d\theta_t^+\left(\mu_B\Gamma^+e^{b\Phi^+/2}+\frac{\mu b^2}{4\mu_B}(\Gamma^-e^{b\Phi^+/2}-\Gamma^+e^{b\Phi^-/2})\right)\Bigg|_{\theta_t^-=0}+\text{c.c.}\right]. \tag{3.35}$$

When the terms including the superfields $\Gamma^{\pm}$ do not exist, Eq. (3.35) reduces to the boundary action which preserves only $N=1$ supersymmetry under Neumann boundary conditions. In this case the $N=2$ supersymmetry transformation of the action $S+S_B$ has a nonvanishing surface term which is canceled by those of the terms including $\Gamma^{\pm}$.

IV. DISCUSSION

Our result contains one boundary parameter μ_B which generates a continuous family of conformal boundary conditions. One remarkable result is that the boundary action has a dual symmetry

$$\mu_B\to\frac{\mu b^2}{4\mu_B}. \tag{4.1}$$

This means that two conformal boundary conditions of the $N=2$ SLFT can be identified. To understand the further implications of this, we need to derive some exact correlation functions such as boundary one-point functions. Our boundary action is a first step toward this. It is possible to derive a functional relation for the one-point functions using the boundary action as a screening boundary operator. The main difficulty arises, as in the bulk case [24], from the lack of coupling constant duality. In a recent paper [25], the one-point functions for the $N=2$ SLFT are conjectured from the modular transformations of the characters for a special value of the coupling constant. It would be interesting to see if these one-point functions are consistent with the functional relations based on our boundary action and to derive them for arbitrary values of the coupling constant.

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APPENDIX A: CONVENTIONS

In this appendix we present our conventions for $N=1,2$ supersymmetries.

1. $N=1$ supersymmetry

We use (1,1) superspace with bosonic coordinates z, $\bar{z}$ and fermionic coordinates θ, $\bar{\theta}$. Here we define $z=x+iy$, $\bar{z}=x-iy$ and $\partial=(\partial_x-i\partial_y)/2$, $\bar{\partial}=(\partial_x+i\partial_y)/2$. The integration measure is $\int d^2zd^2\theta=\int dxdyd\theta d\bar{\theta}$. The covariant derivatives are given by

$$D=\frac{\partial}{\partial\theta}+\theta\partial,\ \bar{D}=\frac{\partial}{\partial\bar{\theta}}+\overline{\theta\partial}, \tag{A1}$$

which satisfy

$$\{D,D\}=2\partial,\ \{\bar{D},\bar{D}\}=2\bar{\partial},\ \{D,\bar{D}\}=0. \tag{A2}$$

The supercharges

$$Q=\frac{\partial}{\partial\theta}-\theta\partial,\ \bar{Q}=\frac{\partial}{\partial\bar{\theta}}-\overline{\theta\partial} \tag{A3}$$

satisfy

$$\{Q,Q\}=-2\partial,\ \{\bar{Q},\bar{Q}\}=-2\bar{\partial},\ \{Q,\bar{Q}\}=0, \tag{A4}$$

and anticommute with D, $\bar{D}$.

2. $N=2$ supersymmetry

We use (2,2) superspace with bosonic coordinates z, $\bar{z}$ and fermionic ones θ^+, $\bar{\theta}^+$, θ^-, $\bar{\theta}^-$. Complex conjugation of fermionic coordinates is defined by $(\theta^{\pm})^*=\bar{\theta}^{\pm}$. The covariant derivatives

$$D_\pm = \frac{\partial}{\partial\theta^\pm} + \frac{1}{2}\theta^\mp\partial,\ \bar{D}_\pm = \frac{\partial}{\partial\bar{\theta}^\pm} + \frac{1}{2}\bar{\theta}^\mp\bar{\partial} \tag{A5}$$

satisfy

$$\{D_+, D_-\} = \partial,\ \{\bar{D}_+, \bar{D}_-\} = \bar{\partial},$$
$$\text{all other (anti)commutators} = 0. \tag{A6}$$

The supercharges are given by

$$Q_\pm = \frac{\partial}{\partial\theta^\pm} - \frac{1}{2}\theta^\mp\partial,\ \bar{Q}_\pm = \frac{\partial}{\partial\bar{\theta}^\pm} - \frac{1}{2}\bar{\theta}^\mp\bar{\partial}, \tag{A7}$$

which obey

$$\{Q_+, Q_-\} = -\partial,\ \{\bar{Q}_+, \bar{Q}_-\} = -\bar{\partial},$$
$$\text{all other (anti)commutators} = 0, \tag{A8}$$

and anticommute with $D_\pm$, $\bar{D}_\pm$. Chiral superfields $\Phi_i^\pm$ satisfy $D_\mp\Phi_i^\pm = \bar{D}_\mp\Phi_i^\pm = 0$. An $N=2$ supersymmetric action is constructed from D terms and F terms[1] and is written as

$$S = \int d^2z d^4\theta K(\Phi_i^+, \Phi_i^-) + \left(\int d^2z d^2\theta^+ W^+(\Phi_i^+) + \text{c.c.}\right), \tag{A9}$$

where $K(\Phi_i^+, \Phi_i^-)$ is an arbitrary differentiable function of superfields and $W^+(\Phi_i^+)$ is a holomorphic function of chiral superfields Φ_i^+. The integration measures in Eq. (A9) are defined by

$$\int d^2z d^4\theta K(\Phi_i^+, \Phi_i^-) = \int dx dy d\theta^+ d\bar{\theta}^+ d\theta^- d\bar{\theta}^- \times K(\Phi_i^+, \Phi_i^-), \tag{A10}$$

$$\int d^2z d^2\theta^+ W^+(\Phi_i^+) = \int dx dy d\theta^+ d\bar{\theta}^+ \times W^+(\Phi_i^+)|_{\theta^- = \bar{\theta}^- = 0}. \tag{A11}$$

APPENDIX B: QUANTUM CORRECTIONS OF THE $N=1$ BOUNDARY SUPERSYMMETRY

In this appendix we show how the classical boundary action (2.10) with Eqs. (2.16) is modified at the quantum level. At the quantum level, we replace $e^{\alpha\phi}$ with the normal-ordered exponential $:e^{\alpha\phi}:$. Therefore, we cannot eliminate $\psi, \bar{\psi}$ from $G, \bar{G}$ because f^{-1} may not be well defined. Therefore, we need to keep the fermionic fields. Then, Eq. (2.15) becomes

$$G - \bar{G} = 2\pi f a \partial_x\phi + 2\pi(\psi+\bar{\psi})\frac{\partial f}{\partial\phi}a(\psi+\bar{\psi}) - 2\pi(\psi+\bar{\psi})\frac{\partial B}{\partial\phi} - 4\pi Q\partial_x f a - 4\pi Q f^2(\psi+\bar{\psi}) + Q\pi\mu b^2(\psi+\bar{\psi})\Lambda^{-2b^2}(:e^{b\phi/2}:)^2, \tag{B1}$$

where Λ is a cutoff scale and $f = \mu_B :e^{b\phi/2}:$.

The first and second terms on the right-hand side of Eq. (B1) can be dealt with the point-splitting technique developed in [26]. The first term can be calculated as

$$\begin{aligned} f a \partial_x\phi &= \frac{\mu_B}{2}\lim_{x_1\to x_2}[:e^{b\phi(x_1)/2}:a(x_1)\partial_x\phi(x_2) + (x_1 \leftrightarrow x_2)] \\ &= \mu_B :e^{b\phi/2}\partial_x\phi: a + b\lim_{x_1\to x_2}\frac{1}{x_1 - x_2} \\ &\quad\times[f(x_1)a(x_1) - f(x_2)a(x_2)] \\ &= \frac{2}{b}\partial_x f a + b\partial_x f a + b f^2(\psi+\bar{\psi}), \end{aligned} \tag{B2}$$

where we used Eq. (2.14). Similarly the second term becomes

$$(\psi+\bar{\psi}) f a (\psi+\bar{\psi}) = 2\partial_x f a + 2 f^2(\psi+\bar{\psi}). \tag{B3}$$

This leads to

$$G - \bar{G} = \pi(\psi+\bar{\psi})\left[\left(-\frac{4}{b}\mu_B^2 + \hat{Q}\mu b^2\Lambda^{-2b^2}\right)(:e^{b\phi/2}:)^2 - 2\frac{\partial B}{\partial\phi}\right], \tag{B4}$$

where we used $\hat{Q} = 1/b + b$. The condition $G - \bar{G} = 0$ gives

$$B = \left(-\frac{2}{b^2}\mu_B^2 + \frac{1}{2}\hat{Q}b\mu\Lambda^{-2b^2}\right)(:e^{b\phi/2}:)^2. \tag{B5}$$

Compared with the classical result (2.16), B gets the quantum correction.

The condition $T - \bar{T} = 0$ can be also satisfied in this way. Substituting Eqs. (2.13) and (2.14) into $T - \bar{T}$, we obtain

$$T - \bar{T} = \pi i[2\partial_x f a(\psi+\bar{\psi}) - \partial_x(f a(\psi+\bar{\psi}))] + \frac{1}{2}(\bar{\psi}\partial_x\bar{\psi} - \psi\partial_x\psi). \tag{B6}$$

Using Eq. (2.14) for ψ and $\bar{\psi}$, one can show that Eq. (B6) vanishes. Therefore, the boundary action (2.10) along with f and B given above preserves boundary conformal symmetry up to the quantum level.

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