

One-point functions of loops and constraint equations of the multi-matrix models at finite N

Changrim Ahn ¹

F.R. Newman Laboratory of Nuclear Studies, Cornell University, Ithaca, NY 14853, USA

and

Kazuyasu Shigemoto ²

Department of Physics, Tezukayama University, Nara 631, Japan

Received 2 January 1992; revised manuscript received 27 April 1992

We derive one-point functions of the loop operators of hermitian matrix-chain models at finite N in terms of differential operators acting on the partition functions. The differential operators are completely determined by recursion relations from the Schwinger–Dyson equations. The interesting observation is that these generating operators of the one-point functions satisfy the $W_{1+\infty}$ like algebra. Also, we obtain constraint equations on the partition functions in terms of the differential operators. These constraint equations on the partition functions define the symmetries of the matrix models at an off-critical point before taking the double scaling limit.

1. Introduction

Recently much progress has been made on the matrix model formulation of the 2D gravity to study the non-perturbative effects [1], and an interesting connection with the integrable systems has been made in the double scaling limit, in which the size of the matrix N becomes infinite and the matrix potentials have critical forms [1]. In this limit, the non-perturbative results can be obtained from non-linear integrable differential equations, such as KdV equations. Furthermore, the correlation functions satisfy their hierarchical equations [2,3].

It has been noticed recently that the integrability of the matrix models is maintained even at off-critical points (finite N) before taking the double scaling limit. At finite N , the Lax pair, zero-curvature conditions, and an infinite number of conserved quantities of the matrix model have been derived and related to integrable systems in a more clear and direct way. The underlying integrable systems have been identified with the 1D Toda hierarchy for the one-matrix model [4] which becomes the KdV hierarchy in the scaling limit [5], the 2D Toda hierarchy [4,6], and the 2D Toda multi-component hierarchy [7] for the two-matrix model and for the general multimatrix models, respectively. The partition functions are the “ τ -functions” of these integrable systems.

The next object of interest is the correlation functions of local operators. For the operators appearing in the action, the correlation functions are simply given by the derivatives of the partition functions with respect to the coefficients of the operators in the action. It requires, however, a non-trivial analysis for the operators which do not appear in the action. In this paper, we derive one-point functions for the general local operators which are the “loops” in the matrix models in terms of differential operators acting on the partition functions [8]. These operators satisfy the recursion relations coming from the Schwinger–Dyson equations. We notice that the

¹ E-mail address: ahn@cornella.bitnet.

² E-mail address: shigemot@jpnrfp.bitnet.

commutation relations of these differential operators are similar to those of the $W_{1+\infty}$ algebra and become exact in the continuum limit.

One important related problem is the symmetry structure of the matrix models. The Virasoro and W_{p+1} algebras have been conjectured for the p multi-matrix models as constraint equations on the partition functions in the double scaling limit [9,10]. The derivation of these symmetries, however, has not been made except for the one-matrix model (the Virasoro algebra) and a special two-matrix model [11] (the W_3 algebra). This derivation may be possible if one considers the constraint equations of the matrix models at finite N first. Indeed, it is at finite N that the Virasoro algebras have been derived for the one-matrix model [5,1,2,13] and for the multi-matrix model [7]. In this paper, we derive most general constraint equations for the multi-matrix models in terms of the generators of the $W_{1+\infty}$ -like algebra. These constraint equations seem to be consistent with the conjectures made in the double scaling limit.

2. Two-matrix model

The partition function of the hermitian two-matrix model is given by

$$Z[\{t_k\}; \{s_k\}, c] = \int \mathcal{D}U \mathcal{D}V e^{-S},$$

$$S = \gamma_1(U) + \gamma_2(V) = cUV, \quad \gamma_1(U) = \sum_{k=1}^{\infty} t_k U^k, \quad \gamma_2(V) = \sum_{k=1}^q s_k V^k. \tag{1}$$

Note the difference in the two potentials γ_1 and γ_2 ; γ_1 is an arbitrary polynomial potential and γ_2 is with fixed order. We want to express correlation functions in terms of t_k and their derivatives acting on the partition functions. These differential operators depend explicitly on the other parameters s_k .

The most interesting loop operators in the two-matrix models are $\text{Tr}(V^n U^m)$. The one-point functions of these loops are

$$\langle \text{Tr}(V^n U^m) \rangle = \int \mathcal{D}U \mathcal{D}V e^{-S} [\text{Tr}(V^n U^m)]. \tag{2}$$

From the Schwinger–Dyson (SD) equations,

$$\sum_{i,j=1}^N \int \mathcal{D}U \mathcal{D}V \frac{\partial}{\partial X_{ij}} [(V^n U^m)_{ij} e^{-S}] = 0 \quad (X=U, V) \quad (m, n \geq 0), \tag{3}$$

one can derive two recursion relations as follows:

$$c \langle \text{Tr}(V^{n+1} U^m) \rangle = \langle \text{Tr}(V^n U^m \gamma_1'(U)) \rangle - \sum_{r=0}^{m-1} \langle \text{Tr}(V^n U^{m-r-1}) \text{Tr} U^r \rangle, \tag{4}$$

$$c \langle \text{Tr}(V^n U^{m+1}) \rangle = \langle \text{Tr}(V^n \gamma_2'(V) U^m) \rangle - \sum_{s=0}^{n-1} \langle \text{Tr} V^s \text{Tr}(V^{n-s-1} U^m) \rangle. \tag{5}$$

Differential operators generating the one-point functions, defined by

$$W_{m-n}^{(n+1)} Z[\{t_k\}; \{s_k\}, c] \equiv -c^n \langle \text{Tr}(V^n U^m) \rangle \tag{6}$$

satisfy the recursion relation from eq. (4),

$$W_{m-n-1}^{(n+2)} = \sum_{r=0}^{m-1} \frac{\partial}{\partial t_r} \circ W_{m-n-r-1}^{(n+1)} + \sum_{r=1}^{\infty} r t_r W_{m-n+r-1}^{(n+1)},$$

$$W_m^{(1)} = \frac{\partial}{\partial t_m}, \quad \frac{\partial}{\partial t_0} \equiv -N, \quad \text{for } m, n \geq 0, \tag{7}$$

where the symbol \circ is defined by $(A \circ B)Z = A(BZ)$. This recursion relation can be rewritten in a simple form,

$$W_m^{(n+1)} = \sum_{r=-\infty}^{m+n-1} \times J_r \circ W_{m-r}^{(n)} \times \quad (n \geq 0, m \geq -n), \quad J_r = \partial/\partial t_r, \quad \text{if } r > 0, \tag{8}$$

$$= r t_r, \quad \text{if } r < 0,$$

where $\times \times$ denotes the normal ordering. The operator J_r satisfy the U(1) current algebra $[J_m, J_n] = m\delta_{m+n,0} (J(z) = \sum_m J_m z^{-m+1} = \partial_z \phi)$. Eq. (8) defines recursively the generating differential operators of the one-point functions. If there is no upper limit in the summation range ($m+n \rightarrow \infty$), it is obvious that

$$W^{(n)}(z) = \sum_m W_m^{(n)} z^{-m+n} = \times (\partial_z \phi)^n \times. \tag{9}$$

These infinite number of currents $W^{(n)}(z)$ ($n = 1, 2, \dots$) generate the $W_{1+\infty}$ algebra [14]. For the finite values of $m+n$, however, the commutation relations are not exactly the same as those of the $W_{1+\infty}$ algebra. This $W_{1+\infty}$ -like algebra generates the one-point functions of the loops. It is remarkable that in the continuum limit the loop operators are given by the operators like $\text{Tr}(U^M)$ with "lattice spacing" $a \rightarrow 0$ and "lattice size" $M \rightarrow \infty$ while keeping aM finite. Therefore, the $W_{1+\infty}$ algebra generates the one-point functions in the double scaling limit.

For explicit examples and later use, we write explicit expressions for $W_m^{(2)}, W_m^{(3)}$,

$$W_m^{(2)} = \sum_{r=0}^m \partial_r \partial_{m-r} + \sum_{r>0} r t_r \partial_{m+r},$$

$$W_m^{(3)} = \sum_{r=0}^{m+1} \sum_{s=0}^{m-r} \partial_r \partial_s \partial_{m-r-s} + \sum_{r>0} r t_r \left(\sum_{s=0}^{m+1} \partial_s \partial_{m+r-s} + \sum_{s=0}^{m+r} \partial_s \partial_{m+r-s} \right) + \sum_{r,s=0}^{\infty} r t_r s t_s \partial_{m+r+s} + \frac{1}{2} (m+2)(m+1) \partial_m, \tag{10}$$

where $\partial_m = \partial/\partial t_m$ and $W_m^{(2)}$ can be identified with the Virasoro generator L_m , as it satisfies the classical Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n}$.

Now consider the constraint equations on the partition function. One can express eq. (5) with the one-point generating operators as follows:

$$\hat{W}_m^{(n+1)} Z[\{t_k\}; \{s_k\}, c] = 0, \quad \text{for } n \geq 0, m \geq -n,$$

$$\hat{W}_m^{(n+1)} = W_m^{(n+1)} - \sum_{k=1}^q \frac{k s_k}{c^k} W_{m-k}^{(k+n)} - \sum_{k=0}^{n-1} W_{m+k}^{(n-k)} \circ W_{-k}^{(k+1)}. \tag{11}$$

Not all of these constraints are independent. In fact, we can prove that only the $\hat{W}_m^{(1)}$ are independent by showing the following relation from eqs. (4) and (11)

$$\hat{W}_m^{(n+1)} = \sum_{r=-\infty}^{m+n-1} \times J_r \circ \hat{W}_{m-r}^{(n)} \times \quad (n \geq 0, m \geq -n). \tag{12}$$

If $\hat{W}_m^{(1)} Z = 0, \hat{W}_m^{(n>1)} Z = 0$ are automatically satisfied. Therefore, the constraint equations for the two-matrix models become

$$\hat{W}_m^{(1)} Z = \left(\frac{\partial}{\partial t_m} - \sum_{k=1}^q \frac{k s_k}{c^k} W_{m-k}^{(k)} \right) Z = 0. \tag{13}$$

As one can see in eq. (13), the constraints are linear combinations of the generators of the $W_{1+\infty}$ -like algebra

with the coefficients of the second potential γ_2 . In the continuum limit, the constraint equations are given by the $W_{1+\infty}$ algebra. Furthermore, for a special potential γ_2 , the operators $\tilde{W}_m^{(1)}$ may generate a subalgebra of the $W_{1+\infty}$, say, the W_n algebra.

One can realize the $W_{1+\infty}$ -like algebra as a symmetry of the matrix model in the context of quantum field theory. In the ordinary quantum field theory, the symmetry can be found as infinitesimal changes of the quantum fields which leave the action invariant. For the matrix model, this can be done by the following generators $A_{m,n}, B_{m,n}$ defined by

$$A_{m,n}[e^{-S}] = \sum_{i,j=1}^N \frac{\partial}{\partial U_{ij}} [(U^m V^n)_{i,j} e^{-S}], \quad B_{m,n}[e^{-S}] = \sum_{i,j=1}^N \frac{\partial}{\partial V_{ij}} [(U^m V^n)_{i,j} e^{-S}]. \tag{14}$$

The generators $A_{m,n}$ and $B_{m,n}$ satisfy the following closed commutation relations:

$$[A_{m,0}, B_{k,l}] + kB_{m+k-1,l}, \quad [A_{m,n}, B_{0,l}] = -nA_{m,n+l-1}, \quad [A_{m,1}, B_{1,l}] = B_{m,l+1} - A_{m+1,l}. \tag{15}$$

Note that $\langle A_{m,n}[e^{-S}] \rangle = \langle B_{m,n}[e^{-S}] \rangle = 0$ from the SD equations (3), which become the constraint equations as shown above. This realization makes it simple to prove the statement that only the $n=1$ constraints are independent. This comes from the fact that $\langle B_{m,n}[e^{-S}] \rangle = 0$ can be obtained from $\langle B_{m,0}[e^{-S}] \rangle = 0$ by using the commutation relations (15). In fact, it is not difficult to see that $\{A_{0,0}, A_{2,1}, B_{1,0}, B_{0,2}\}$ are enough to generate the constraints.

3. Multi-matrix models

The multi-matrix models with p hermitian matrix variables U_a have the partition function

$$Z[\{t_k\}; \{s_{a,k}\}, \{c_a\}] = \int \prod_{a=1}^p \mathcal{D}U_a e^{-S}, \quad S = \text{Tr} \left(\sum_{a=1}^p \gamma_a(U_a) - \sum_{a=1}^{p-1} c_a U_a U_{a+1} \right). \tag{16}$$

We will choose the matrix potentials

$$\gamma_1(U_1) = \sum_{k=1}^{\infty} t_k U_1^k, \quad \gamma_a(U_a) = \sum_{k=1}^{q_a} s_{a,k} U_a^k, \tag{17}$$

considering the t_k as variables and the $s_{a,k}$ as fixed parameters. The loop operators in the multi-matrix models are given by $\text{Tr}[U_p^{n_p} \dots U_2^{n_2} U_1^{n_1}]$. Again, we want to express one-point functions of these operators in terms of the linear differential operators.

We start with the SD equations:

$$\begin{aligned} \sum_{i,j=1}^N \int [\mathcal{D}U] \frac{\partial}{\partial (U_1)_{ij}} [(U_1^{n_1} U_p^{n_p} \dots U_2^{n_2})_{ij} e^{-S}] &= 0, \\ \sum_{i,j=1}^N \int [\mathcal{D}U] \frac{\partial}{\partial (U_a)_{ij}} [(U_{a-1}^{n_{a-1}} \dots U_1^{n_1} U_p^{n_p} \dots U_{a+1}^{n_{a+1}})_{ij} e^{-S}] &= 0 \quad (2 \leq a \leq p-1), \\ \sum_{i,j=1}^N \int [\mathcal{D}U] \frac{\partial}{\partial (U_p)_{ij}} [(U_p^{n_p-1} \dots U_1^{n_1} U_p^{n_p})_{ij} e^{-S}] &= 0. \end{aligned} \tag{18}$$

Eq. (18) can be used to derive recursion relations for the one-point functions of the loop operators:

$$c_1 \langle \text{Tr}(U_p^{n_p} \dots U_2^{n_2+1} U_1^{n_1}) \rangle = - \sum_{r=0}^{n_1-1} \langle \text{Tr} U_1^r \text{Tr}(U_p^{n_p} \dots U_2^{n_2} U_1^{n_1-r-1}) \rangle + \langle \text{Tr}(U_p^{n_p} \dots U_2^{n_2} U_1^{n_1} \gamma_1'(U_1)) \rangle, \tag{19}$$

$$c_{a-1} \langle \text{Tr}(U_p^{n_p} \dots U_{a+1}^{n_{a+1}} U_{a-1}^{n_{a-1}+1} \dots U_1^{n_1}) \rangle + c_a \langle \text{Tr}(U_p^{n_p} \dots U_{a+1}^{n_{a+1}+1} U_{a-1}^{n_{a-1}} \dots U_1^{n_1}) \rangle$$

$$= \langle \text{Tr}(U_p^{n_p} \dots U_{a+1}^{n_{a+1}} \mathcal{V}'_a(U_a) U_{a-1}^{n_{a-1}} \dots U_1^{n_1}) \rangle, \quad 2 \leq a \leq p-1, \quad (20)$$

$$c_{p-1} \langle \text{Tr}(U_p^{n_p} U_{p-1}^{n_{p-1}+1} \dots U_1^{n_1}) \rangle = - \sum_{r=0}^{n_p-1} \langle \text{Tr} U_p^r \text{Tr}(U_p^{n_p-r-1} U_{p-1}^{n_{p-1}} \dots U_1^{n_1}) \rangle$$

$$+ \langle \text{Tr}(\mathcal{V}'_p(U_p) U_p^{n_p} U_{p-1}^{n_{p-1}} \dots U_1^{n_1}) \rangle. \quad (21)$$

Eq. (19) can be rewritten in the form of the recursion relations in terms of the differential operators

$$\mathcal{W}_m^{(n+1)}(n_3, \dots, n_p) = \sum_{r=-\infty}^{m+n-1} \times J_r \circ \mathcal{W}_{m-r}^{(n)}(n_3, \dots, n_p) \times,$$

$$\mathcal{W}_{m-n}^{(n+1)}(n_3, \dots, n_p) Z[\{t_k\}] = -c_1^n c_2^{n_3} \dots c_{p-1}^{n_p} \langle \text{Tr}(U_p^{n_p} \dots U_2^{n_2} U_1^{n_1}) \rangle. \quad (22)$$

We want to show that any one-point function can be expressed in terms of the variables t_k and their derivatives. Since eq. (22) for $n_3 = \dots = n_p = 0$ with the identification $c = c_1$ is exactly the same as eq. (8), $\mathcal{W}_m^{(n+1)}(0, \dots, 0) = W_m^{(n+1)}$ of the two-matrix model.

To derive other one-point functions, we consider other recursion formulae coming from eq. (20) ($2 \leq a \leq p-1$)

$$\mathcal{W}_m^{(n+1)}(n_3, \dots, n_{a-1}, 0, n_{a+1}+1, \dots, n_p)$$

$$= -\frac{c_{a-1}}{c_{a-2}} \mathcal{W}_m^{(n+1)}(n_3, \dots, n_{a-1}+1, 0, n_{a+1}, \dots, n_p) + \sum_{k=1}^{q_a} \frac{k s_{a,k}}{c_{a-1}^{k-1}} \mathcal{W}_m^{(n+1)}(n_3, \dots, n_{a-1}, k-1, n_{a+1}, \dots, n_p). \quad (23)$$

Assuming that we can express all operators $\mathcal{W}_m^{(n)}(0, \dots, 0, n_a, n_{a+1}, \dots, n_p)$ in terms of t_k , we can find $\mathcal{W}_m^{(n)}(0, \dots, 0, n_{a+1}, \dots, n_p)$ by repeatedly using eq. (23). Therefore, we showed inductively that all the one-point functions can be found as differential operators of the t_k acting on the partition function. Finally, if one finds all the operators in terms of the t_k , one can find the constraint equations on the partition function from eq. (21). Among others, the case of $n_p = 0$ gives the following equations:

$$\hat{W}_m^{(1)}(n_3, \dots, n_{p-1}+1, 0) Z[\{t_k\}; \{s_{a,k}\}, \{c_a\}] = 0,$$

$$\hat{W}_m^{(1)}(n_3, \dots, n_{p-1}+1, 0) = \mathcal{W}_m^{(1)}(n_3, \dots, n_{p-1}+1, 0) - c_{p-2} \sum_{k=1}^{q_p} \frac{k s_{p,k}}{c_{p-1}^{k-1}} \mathcal{W}_m^{(1)}(n_3, \dots, n_{p-1}, k-1) \quad (p \geq 4) \quad (24)$$

and we must treat carefully the index n in eqs. (23) and (24) for $i \leq 3$.

We apply the above general analysis to the three-matrix model. Defining

$$\mathcal{W}_{m-n}^{(n+1)}(l) Z[\{t_k\}] = -c_1^n c_2^l \langle \text{Tr}(W^l V^n U^m) \rangle, \quad (25)$$

they satisfy the following recursion relations:

$$\mathcal{W}_m^{(n+1)}(l) = \sum_{r=-\infty}^{m+n-1} \times J_r \circ \mathcal{W}_{m-r}^{(n)}(l) \times, \quad \mathcal{W}_m^{(1)}(l+1) = -c_1 \mathcal{W}_{m+1}^{(1)}(l) + \sum_{k=2}^{q_2} \frac{k s_{2,k}}{c_1^{k-1}} \mathcal{W}_m^{(k)}(l). \quad (26)$$

As explained above, from $\mathcal{W}_m^{(n)}(0) = W_m^{(n)}$ one can find the $\mathcal{W}_m^{(1)}(1)$ from the second equation and the $\mathcal{W}_m^{(n)}(l)$ from the first one. Continuing this step, one can find all the $\mathcal{W}_m^{(n)}(l)$. Finally, the constraint equations come from eq. (24). For an explicit example, consider the following potentials: $\mathcal{V}_2(V) = v_2 V^2 + v_3 V^3$ and $\mathcal{V}_3(W) = w_2 W^2 + w_3 W^3$. From eq. (26), one can find the explicit expressions

$$\mathcal{W}_m^{(1)}(1) = -c_1 W_{m+1}^{(1)} + \frac{2v_2}{c_1} W_m^{(2)} + \frac{3v_3}{c_1^2} W_m^{(3)}, \quad (27)$$

$$\begin{aligned} \mathcal{W}_m^{(2)}(1) &= \sum_{r=0}^m \frac{\partial}{\partial t_r} \circ \mathcal{W}_{m-r}^{(1)}(1) + \sum_{r=1}^{\infty} r t_r \mathcal{W}_{m+r}^{(1)}(1), \\ \mathcal{W}_m^{(3)}(1) &= \sum_{r=0}^{m+1} \frac{\partial}{\partial t_r} \circ \mathcal{W}_{m-r}^{(2)}(1) + \sum_{r=1}^{\infty} r t_r \mathcal{W}_{m+r}^{(2)}(1), \\ \mathcal{W}_m^{(1)}(2) &= -c_1 \mathcal{W}_{m+1}^{(1)}(1) + \frac{2v_2}{c_1} \mathcal{W}_m^{(2)}(1) + \frac{3v_3}{c_1^2} \mathcal{W}_m^{(3)}(1), \end{aligned} \tag{27 cont'd}$$

and the constraint equations are

$$\hat{\mathcal{W}}_{m-1}^{(2)}(0)Z = \left(W_{m-1}^{(2)} - \frac{2c_1 w_2}{c_2^2} \mathcal{W}_m^{(1)}(1) - \frac{3c_1 w_3}{c_3^2} \mathcal{W}_m^{(1)}(2) \right) Z = 0, \tag{28}$$

where $W_m^{(2)}$, $W_m^{(3)}$ are given in eq. (10).

4. Discussions

In this paper, we computed one-point functions of the multi-matrix models in terms of the differential operators acting on the partition functions. The operators are completely determined by the recursion formulae, derived from the SD equations and generate the $W_{1+\infty}$ -like algebra. Furthermore, we derived the constraint equations on the partition functions using these operators. Since the partition functions are the “ τ functions” of the 2D Toda hierarchies [4,6,7], this means the one-point functions as well as the partition functions are completely determined by the integrable systems and symmetry structures. Our method can be generalized to the multi-point functions. Again, the generating operators are determined by the recursion relations which come from the SD equations.

It is also possible to consider the s_k as variables such that one can introduce another differential operators for the two-matrix model. For the potentials $\mathcal{V}_1(U) = \sum_k t_k U^k$, $\mathcal{V}_2(V) = \sum_k s_k V^k$, one can define

$$W_{m-n}^{(n+1)} Z[\{t_k\}, \{s_k\}] \equiv -c^n \langle \text{Tr}(V^n U^m) \rangle, \quad \bar{W}_{m-n}^{(n+1)} Z[\{t_k\}, \{s_k\}] \equiv -c^n \langle \text{Tr}(U^n V^m) \rangle, \tag{29}$$

where both $W_m^{(n)}$ and $\bar{W}_m^{(n)}$ satisfy the recursion relation (8). The constraint equations are just

$$[c^{mj} W_{m-n}^{(n+1)} - c^n \bar{W}_{n-m}^{(m+1)}] Z[\{t_k\}, \{s_k\}] = 0. \tag{30}$$

Another interesting point we want to mention is that the constraint equations for the multi-matrix models like eq. (28) have a very similar form as those of the two-matrix models (24). The coefficients of the second potential of the two-matrix model can be decided on by those of the multi-matrix potentials. The correspondence, however, is not quite exact. There appear some terms in the constraint equations of the multi-matrix models which do not exist in those of the two-matrix model. This observation reminds us of the recent conjectures that all the multi-critical points can be achieved by the two-matrix model [15]. If the extra terms at finite N are suppressed in the double scaling limit, our observation can be a proof of this claim. Related to this and other motivations, it would be very interesting to consider the double scaling limit of our formalism. Our discovery that the correlation functions are generated by the $W_{1+\infty}$ algebra acting on the “ τ -functions” of the 2D Toda hierarchy seems to be consistent with the results in the double scaling limit in that the correlation functions of the one-matrix model are given by KdV-hierarchy equations [2] and that the τ -function of the p reduced KP-hierarchy satisfies the W_{1+p} constraint equations [10].

Recently, there have been several papers which mention the $W_{1+\infty}$ algebra. Our $W_{1+\infty}$ algebra is different from that of ref. [16] in that the latter comes from the higher order terms under the change of M to $M + \delta M$. Therefore, this constraint exists even for the one-matrix model. Our $W_{1+\infty}$ constraints exist only for the multi-matrix models and will have a direct connection with the W_n algebra structures conjectured in ref. [9] in the

double scaling limit. Also, the $W_{1+\infty}$ algebra appears from the KP-hierarchy in the double scaling limit [17]. This is a direct $p \rightarrow \infty$ limit of the W_{1+p} constraint equations considered in ref. [10]. It would be interesting to consider the $p \rightarrow \infty$ limit of our result to understand these results from the matrix model point of view.

Note added

While typing this paper, we received a paper [18] from Y.-X. Cheng where constraint equations for the two-matrix model like eq. (30) have been obtained.

Acknowledgement

C.A. thanks the Tezukayama University and the Yukawa Institute of Kyoto University at Uji for their hospitality and financial support where part of this work has been done.

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