$gl(n|m)$ color Calogero-Sutherland models and super Yangian algebra

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Abstract

A supersymmetric extension of the color Calogero-Sutherland model is considered based on the Yangian $Y(gl(n|m))$. The algebraic structure of the model is discussed in some detail. We show that the commuting conserved quantities can be generated from the super-quantum determinant, thus establishing the integrability of the model. In addition, the rational limit of the model is studied where the Yangian symmetry degenerates into a super loop algebra.

1. Introduction

The Calogero-Sutherland model and its $gl(n)$ extension [1] (collectively called CS model here) are integrable models based on a different kind of mechanism for their integrability. Unlike integrable models with short range interactions, the CS model's Hamiltonian and the full set of commuting conserved quantities cannot be obtained from the trace of the T-matrix that gives the generators of the quantum algebra. This is because the CS model has a Yangian symmetry [2] even with finite number of particles and the trace of the T-matrix does not commute with the Yangian algebra generators. Instead it turns out that the generating function for the commuting conserved quantities is the quantum determinant [3]. Obvious as it may seems, this is not true however for the case of the Haldane-Shastry (HS) spin chain [4], which is an integrable spin chain with long range interaction and a Yangian symmetry. The quantum determinant in this case is proportional to the identity operator. So far it is not

clear what the generating function for the commuting conserved quantities of the HS spin chain should be, other than using the approach of [5].

Restricting ourselves to the CS model, the algebraic structure of the model can be summarized as follows [3,6]: The model constitutes an N-body quantum mechanical system; the particles are described by their positions x_j, $j = 1, \ldots, N$ and their n-color degrees of freedom labeled by a_j. The configuration space furnishes a representation for the Yangian $Y(gl(n))$; one starts with N copies of the fundamental representation of $gl(n)$ built from the color indices of the particles, which can be taken to be the set of matrix unit X_j^{ab}; $a, b = 1, \ldots, n$. Then construct the Yangian generators T_0^{ab}, T_1^{ab} using this representation of $gl(n)$ and the variables x_j as follows:

$$T_0^{ab} = \sum_{j=1}^{N} X_j^{ab} \tag{1.1}$$

$$T_1^{ab} = \sum_{j,k=1}^{N} X_j^{ab} \left(\delta_{jk} \frac{\partial}{\partial x_j} + \lambda \theta_{jk} P_{jk} \right) \qquad (1.2)$$

where θ_{jk} is some $\mathbf{C}$ number function that depends only on $x_j - x_k (\neq 0)$, and P_{jk} is an operator that exchanges the color degrees of freedom of particles j, k. It has the expression

$$P_{jk} = \sum_{a,b=1}^{n} X_j^{ab} X_k^{ba}$$

in terms of the matrix unit.

Requiring that T_0^{ab}, T_1^{ab} satisfy the Yangian relations encoded in the following equation:

$$R_{oo'}(u - v) T_0(u) T_{0'}(v)$$
$$= T_{0'}(v) T_0(u) R_{oo'}(u - v) \qquad (1.3)$$

where

$$R_{oo'}(u) = u\mathbf{1} + \lambda P_{oo'} \qquad (1.4)$$

with o, o' denoting the two n-dimensional auxiliary spaces, and

$$T_o(u) = \sum_{a,b=1}^{n} X_o^{ba} \left(\mathbf{1}\delta_{ab} + \sum_{s \geq 0} u^{-s-1} T_s^{ab} \right) \qquad (1.5)$$

(similarly for $T_{o'}(v)$,) one deduces that θ_{jk} has to satisfy the relation

$$\theta_{jk}\theta_{jn} + \theta_{jk}\theta_{nk} - \theta_{jn}\theta_{nk} = \theta_{jk} ; \qquad j \neq k \neq n \neq j$$
$$\theta_{jk} + \theta_{kj} = 1 ; \qquad j \neq k \qquad (1.6)$$

The rest of the Yangian generators T_s^{ab}, $s > 1$ can be generated from T_0^{ab}, T_1^{ab} using the defining relation (1.3), one can then compute the quantum determinant defined as

$$\mathrm{qdet}\, T(u) = \sum_{\sigma \in S_n} (-1)^{\mathrm{sign}(\sigma)} T^{1\sigma(1)}(u) T^{2\sigma(2)}(u + \lambda)$$
$$\times \cdots T^{n\sigma(n)}(u + (n - 1)\lambda) \qquad (1.7)$$

The quantum determinant generates the center of the Yangian algebra and expanding it as formal power series in u^{-1} as

$$\mathrm{qdet}\, T(u) = 1 + u^{-1} Q_0 + u^{-2} Q_1 + u^{-3} Q_2 + \cdots$$
$$(1.8)$$

the operators Q_j's commute with the Yangian generators and therefore among themselves. Moreover, the Hamiltonian

$$H = \sum_{j,k=1}^{N} \left(\delta_{jk} \frac{\partial^2}{\partial x_j^2} + \frac{\lambda(\lambda - P_{jk})}{\sin^2(x_j - x_k)} \right) \qquad (1.9)$$

can be obtained from Q_2. So the rest of the operators Q_j's are conserved quantities. In particular, Q_0 and Q_1 contain the particle number operator (which is N here) and the total momentum operator.

The aim of this paper is to show that one can extend the $gl(n)$ color CS model to the $gl(n|m)$ case, i.e. the particles now carry color indices which form a representation of the $gl(n|m)$ graded Lie algebra. The model in the simplest case where $n = m = 1$ and in the 'rational limit' (see later) is the model studied in [7] under the name supersymmetric CS model without the harmonic potential, for this reason we also call this $gl(n|m)$ extension the supersymmetric CS model even though in this case the Hamiltonian can not be written as a commutator of some supersymmetric charges $Q, Q^\dagger$ with the property $(Q^{(\dagger)})^2 = 0$. In general, the supersymmetric CS model has a Yangian symmetry $Y(gl(n|m))$, which is the Yangian deformation of the universal enveloping algebra of the super (graded) Lie algebra $gl(n|m)$, and algebraic structure described above still prevails in this extension. We give an explicit construction of the representation of the supersymmetric Yangian generators based on this model and show that the Hamiltonian has exactly the same form as that of the CS model, except that the exchange operator now is built out of fundamental representation of the $gl(n|m)$ generator which has the property that when exchanging two fermionic colors an additional factor (-1) is produced. The generator of the commuting conserved quantities in this case is the supersymmetric generalization of the quantum determinant.

The construction of the SUSY CS model is given in detail in Section 3 after an elementary reminder of the Yangian $Y(gl(n|m))$ in Section 2. We then consider in Section 4 various limiting cases of the model, which include the rational limit where the Yangian symmetry degenerates into the loop algebra symmetry. Section 5 contains a discussion and the conclusion.

2. The supersymmetric Yangian

Consider the graded Lie algebra $gl(n|m)$ defined by the set of generators e^{ab}, $a, b = 1, \cdots, n + m$ that satisfy the following relation:

$$
[e^{ab}, e^{cd}\} = \delta_{bc} e^{ad} \\
- (-1)^{(p(a)+p(b))(p(c)+p(d))} \delta_{ad} e^{cb} \quad (2.1)
$$

where $p(a)$ is the $\mathbf{Z}_2$ grading defined as

$$
p(a) = \begin{cases} 0 & a = 1, \cdots, n \\ 1 & a = n + 1, \cdots, n + m \end{cases}
$$

and $[\; , \; \}$ is the graded Lie bracket.

Let V be an $n + m$ dimensional $\mathbf{Z}_2$ graded vector space and $\{v^a, a = 1, \cdots, n + m\}$ be a homogeneous basis whose grading is given as above. The matrix unit X^{ab} defined as

$$
X^{ab} v^c = \delta_{bc} v^a \quad (2.2)
$$

furnishes a vector representation ρ for $gl(n|m)$ as follows:

$$
\rho(e^{ab}) = X^{ab} \quad (2.3)
$$

$$
[X^{ab}, X^{cd}\} \equiv X^{ab} X^{cd} \\
- (-1)^{(p(a)+p(b))(p(c)+p(d))} X^{cd} X^{ab} \quad (2.4)
$$

where the second line defines the graded bracket.

In the sequel, we shall refer to the vector v^a with grading $p(a) = 0(1)$ as the bosonic- (fermionic-) color state, and the term supersymmetric arises from the fact that element e^{ba} with $p(a) \neq p(b)$ changes the nature of the color state from bosonic to fermionic or vice versa.

One can construct from these matrix units a representation of the symmetric group $\mathbf{S}_N$. For this purpose, consider N copies of the matrix unit X_j^{ab}, $j = 1, \cdots, N$ that act on the graded vector space $V_1 \otimes \cdots \otimes V_N$ where the subscript j corresponds to the space $V_j \cong V$ in the tensor product. With the relation

$$
X_j^{ab} X_k^{cd} = (-1)^{(p(a)+p(b))(p(c)+p(d))} X_k^{cd} X_j^{ab} \quad (2.5)
$$

one can show that the operator defined as

$$
P_{jk} = \sum_{a,b=1}^{n+m} (-1)^{p(b)} X_j^{ab} X_k^{ba} \quad (2.6)
$$

exchanges the basis vectors v_j, v_k of the j, k space as

$$
P_{jk} v_j^a \otimes v_k^b = (-1)^{p(a)p(b)} v_j^b \otimes v_k^a \quad (2.7)
$$

and satisfies the relations

$$
P_{ij} = P_{ji}, \quad P_{ij} P_{ij} = 1, \quad P_{ij} P_{jk} = P_{ik} P_{ij} \quad (2.8)
$$

Hence it furnishes a representation for the symmetric group $\mathbf{S}_N$

Using this supersymmetric representation for the symmetric group it is straight forward to construct the Yangian deformation of $Y(gl(n|m))$. The $R_{oo'}(u)$ matrix acting on $V_o \otimes V_{o'}$ has the same expression as (1.4), except that the exchange operator $P_{oo'}$ has a representation given in (2.6) with j, k replaced by o, o' that denote the auxiliary spaces, respectively. Similarly let $\{T_s^{ab}, s \geq 0, a, b = 1, \cdots, n + m\}$ be the Yangian $Y(gl(n|m))$ generators and define the T-matrix as

$$
T_o(u) = \sum_{a,b=1}^{n} X_o^{ba} (-1)^{p(a)} T^{ab}(u) \quad (2.9)
$$

where

$$
T^{ab}(u) \equiv \mathbf{1} \delta_{ab} (-1)^{p(a)} + \lambda \sum_{s \geq 0} u^{-s-1} T_s^{ab} \quad (2.10)
$$

The Yangian relation is then encoded in (1.3). This defining relation written in terms of T_s^{ab} is given by

$$
[T_s^{dc}, T_{p+1}^{ba}\} - [T_{p+1}^{dc}, T_s^{ba}\} \\
= \lambda (-1)^{(p(b)p(d)+p(c)p(b)+p(c)p(d))} \\
\times (T_p^{bc} T_s^{da} - T_s^{bc} T_p^{da}) \quad (2.11)
$$

for $s, p \geq -1$ where $T_{-1}^{ab} \equiv \lambda^{-1} \mathbf{1} \delta_{ab}$ and the graded bracket here has the same definition as that in (2.4).

The center of the Yangian algebra can likewise be expressed in a supersymmetric generalized quantum determinant [8]. However, for our purpose we employ an alternative expression for the generator of the center elements. Define the $\tilde{T}_s^{ab}$ operators by the following condition [8]:

$$
\sum_b (-1)^{p(b)} T^{ab}(u) \tilde{T}^{bc}(u) = (-1)^{p(a)} \delta_{ac} \quad (2.12)
$$

where $\tilde{T}^{ab}(u)$ is defined in terms of $\tilde{T}_s^{ab}$ as in (2.10). This gives, for example, the first few modes of $\tilde{T}^{bc}(u)$

$$
\tilde{T}_0^{bc} = -T_0^{bc}
$$

$$\tilde{T}_1^{bc} = -T_1^{bc} + \lambda \sum_f (-1)^{p(f)} T_0^{bf} T_0^{fc}$$

$$\tilde{T}_2^{bc} = -T_2^{bc} + \lambda \sum_f (-1)^{p(f)} (T_0^{bf} T_1^{fc} + T_1^{bf} T_0^{fc})$$

$$- \lambda^2 \sum_{f,g} (-1)^{p(f)+p(g)} T_0^{bf} T_0^{fg} T_0^{gc} \qquad (2.13)$$

The generator $Z(u)$ of the center is then given by

$$Z(u) = \begin{cases} \displaystyle\sum_{a,b} (-1)^{p(b)} T^{ab}(u+(n-m)\lambda) \\ \qquad \times \tilde{T}^{ba}(u)/(n-m) \ ; \qquad n \neq m \\[2mm] 1 + \displaystyle\sum_{a,b} (-1)^{p(b)} \\ \qquad \times \dfrac{d}{du}(T^{ab}(u))\tilde{T}^{ba}(u) \ ; \qquad n = m \end{cases} \qquad (2.14)$$

Expanding $Z(u)$ in power series of u^{-1}, we get

$$Z(u) = 1 + \lambda u^{-2} Z_0 + 2\lambda u^{-3} Z_1 + 3\lambda u^{-4} Z_2$$
$$+ \cdots \qquad (2.15)$$

where

$$Z_0 = Q_0$$
$$Z_1 = Q_1 + \tfrac{1}{2}(n-m)\lambda Q_0$$
$$Z_2 = Q_2 + (n-m)\lambda Q_1 + \tfrac{1}{3}(n-m)^2 \lambda^2 Q_0$$

with

$$Q_0 = \sum_a T_0^{aa}$$

$$Q_1 = \sum_a T_1^{aa} - \tfrac{1}{2}\lambda \sum_{a,b}(-1)^{p(b)} T_0^{ab} T_0^{ba}$$

$$Q_2 = \sum_a T_2^{aa} - \lambda \sum_{a,b}(-1)^{p(b)} T_0^{ab} T_1^{ba}$$

$$+ \tfrac{1}{3}\lambda^2 \sum_{a,b,c}(-1)^{(p(b)+p(c))} T_0^{ab} T_0^{bc} T_0^{ca}$$

$$- \tfrac{1}{6}\lambda(n-m) \sum_{a,b}(-1)^{p(b)} T_0^{ab} T_0^{ba} \qquad (2.16)$$

which are the first few central elements of interest to us. Notice that unlike the mode expansion of the quantum determinant, here the coefficient of u^{-1} vanishes.

It is clear that the special case $n = 0(m = 0)$ corresponds to the Yangian $Y(gl(n))$ with deformation parameter $\lambda(-\lambda)$. From (2.11) we see that interchanging the grading of the basis vector to

$$p(a) = \begin{cases} 1 & a = 1, \cdots, n \\ 0 & a = n+1, \cdots, n+m \end{cases}$$

corresponds to changing λ to $-\lambda$. Physically this will correspond to interchanging the bosonic and fermionic color.

In the next section we will study a particular realization of this supersymmetric Yangian which has physical interest.

3. $gl(n|m)$ color Calogero-Sutherland model

Consider a collection of N identical particles described by the set of coordinates $\{x_j\}$ and carry color index $a = 1, \cdots, n + m$. By definition, the particles are called bosons (fermions) if the wavefunction is symmetric (anti-symmetric) under exchange of the particles (i.e. coordinates and color indices). As mentioned in previous section, depending on the value of the color index, the color degrees of freedom can have a bosonic or fermionic character according to the corresponding Z_2 grading, in order not to confuse this fermionic/bosonic nature of the color with fermionic/bosonic identity of the particle (as defined by the property of the wave function under exchange operation), we shall always refer to the former as fermionic/bosonic-color. For definiteness, restrict to the case of N identical bosons, the configuration space is given by $\mathbf{Sym}(V^{\otimes N} \otimes \mathbf{C}[x_1, \cdots, x_N])$ where $\mathbf{C}[x_1, \cdots, x_N]$ denotes the set of complex functions in the variable $\{x_1, \cdots, x_N\}$ and $\mathbf{Sym}$ the projection onto the subspace which is symmetric under simultaneous exchange of x_j, x_k and v_j, v_k for any j, k. Note that such restriction is justified as long as the the Hamiltonian is invariant under this operation. In general, for a mixture of fermions and bosons, one has to consider projection onto the subspace which corresponds to the Young tableau that describes the exchange properties of the collection of the particles.

From the configuration space, one can construct a representation of the supersymmetric Yangian. To start, we define the action of T_0^{ab} and T_1^{ab} on this space as follows:

$$T_0^{ab} = \sum_{j=1}^{N} X_j^{ab} \tag{3.1}$$

$$T_1^{ab} = \sum_{j,k=1}^{N} X_j^{ab} L_{jk} \tag{3.2}$$

where

$$L_{jk} = \delta_{jk} \frac{\partial}{\partial x_j} + \lambda (1 - \delta_{jk}) \theta_{jk} P_{jk}$$

is an operator that plays the role of one of the Lax pair element in [9], and X_j^{ab}, P_{jk} have been defined in the previous section, while the function $\theta_{jk} \equiv \theta(x_j - x_k)$ is yet to be determined.

Next, we require that these 0, 1 modes of the T-matrix satisfy the Yangian relation (2.11). It is easy to check that (2.11) hold for $s, p = -1, 0$. For other values of s, p, the Yangian relation can be considered as a relation that generates higher mode of the T-matrix from the lower modes ones. Requiring that the T-matrix generated be consistent imposes conditions on the function θ_{jk}. Consider, for example, $s = 1, p = 0$ in (2.11), we get

$$[T_1^{ba}, T_1^{dc}] = \delta_{ad} T_2^{bc} - (-1)^{(p(a) \cdot p(b))(p(c) + p(d))} T_2^{da}$$
$$+ \lambda (-1)^{(cd + a(c+d))} \left(T_0^{bc} T_1^{da} - T_1^{bc} T_0^{da} \right) \tag{3.3}$$

substituting the representation of T_0^{ab}, T_1^{ab} into the above, we find that the function θ_{jk} satisfies exactly the same relations given in (1.6), the general solution is therefore given by

$$\theta_{jk} = (1 - a^{x_j - x_k})^{-1} \tag{3.4}$$

where a is an arbitrary $\mathbf{C}$ number. In addition, the above also gives

$$T_2^{ab} = \sum_{j,k=1}^{N} X_j^{ab} (L^2)_{jk} \tag{3.5}$$

Similarly, we find

$$T_3^{ab} = \sum_{j,k=1}^{N} X_j^{ab} (L^3)_{jk} \tag{3.6}$$

and one can check that (1.6) is the necessary condition for T_3^{ab} to be generated consistently. In an alternative definition of the Yangian algebra [2] only $\{T_s^{ab}; s =$

$0, 1, 2, 3\}$ are needed. Hence, it suffices to check the above construction up to T_3^{ab}.

It is no coincidence that the function θ_{jk} satisfies the same relation as that of the CS model. To see this, note that since the wavefunction is invariant under exchange of two particles, one can trade the colors exchange operator P_{jk} with the coordinates exchange operator K_{jk} and following the approach of [3] to show that

$$T^{ab}(u) = \mathbf{1}\delta_{ab} + \sum_{j=1}^{N} \frac{X_j^{ab}}{u - D_j} \tag{3.7}$$

with

$$D_j = \frac{\partial}{\partial x_j} + \lambda \sum_{k \neq j} \theta_{jk} K_{jk}$$

satisfies the Yangian defining relation. Furthermore, this also implies that

$$T_s^{ab} = \sum_{j,k} X_j^{ab} (L^s)_{jk} ; \quad s \geq 0 \tag{3.8}$$

The same argument can also be applied to the case when all particles are fermions or even mixture of fermions and bosons. In which case the P_{jk} is replaced with $\pm K_{jk}$ depending on the bosonic/fermionic identities of particles j, k, and the proof used in [3] still hold.

With the representation for T_s^{ab} we can compute the central elements of the supersymmetric Yangian. The first few central elements in (2.16) become

$$Q_0 = N \tag{3.9}$$

$$Q_1 = P - \tfrac{1}{2} \lambda N(n - m) \tag{3.10}$$

$$Q_2 = 2H - \lambda(n - m)P - \tfrac{1}{6} \lambda N(N - 1)$$
$$+ \tfrac{1}{6} \lambda^2 (n - m)^2 N \tag{3.11}$$

where P, H are the total momentum and Hamiltonian of the system of particles given by

$$P = \sum_j \frac{\partial}{\partial x_j} \tag{3.12}$$

$$H = \frac{1}{2} \sum_{j,k} \left(\partial_j^2 + \lambda P_{jk} \partial_j \theta_{jk} + \lambda^2 \theta_{jk} \theta_{kj} \right) \tag{3.13}$$

So from this construction, it is clear that the rest of the central elements, which commute with H, are

conserved quantities of the model and they commute among themselves, so the model is integrable. Using (3.4), the Hamiltonian can be written explicitly

$$H = \frac{1}{2} \sum_j \frac{\partial^2}{\partial x_j^2} + \sum_{j>k} \frac{\lambda(\lambda - P_{jk})}{\sinh^2[\frac{a}{2}(x_j - x_k)]} \quad (3.14)$$

Notice indeed that the Hamiltonian is invariant under simultaneous permutation of particles' coordinates and color indices, hence the justification of restricting the Hilbert space to be the subspace of $V^{\otimes N} \otimes \mathbf{C}[x_1, \cdots, x_N]$ which are irreducible module of the symmetric group $\mathbf{S}_N$. As T_s^{ab} leave the Hilbert space invariant, the action of, say, $T_s^{ab} T_p^{ab}$ on a totally symmetric wavefunction (i.e. N identical Bosons case) to remain symmetric even though for $p(a) \neq p(b)$ some of the color degrees of freedom will be changed from bosonic to fermionic or vice versa. On the other hand, if two bosonic particles j, k have fermionic colors in the above symmetric wavefunction, letting the quantum numbers of these two particles to be equal, we see that the wave function has to vanish.

4. Special limits of the SUSY CS model

First we study the rational limit of the model. This can be achieved by rescaling $\lambda = a\lambda'$ and letting $a \to 0$. The function θ_{jk} becomes ω_{jk} defined as

$$\omega_{jk} = \frac{1}{x_j - x_k} \quad (4.1)$$

while the Hamiltonian becomes

$$H = \frac{1}{2} \sum_{j,k} \left(\partial_j^2 + \lambda' P_{jk} \partial_j \omega_{jk} + \lambda'^2 \omega_{jk} \omega_{kj} \right) \quad (4.2)$$

The L_{jk} operator remains non-vanishing as

$$L_{jk} = \delta_{jk} \frac{\partial}{\partial x_j} + \lambda' \omega_{jk} P_{jk} \quad (4.3)$$

but the Yangian relation, under this rescaling becomes

$$\left[T_m^{dc}, T_{n+1}^{ba} \right] - \left[T_{m-1}^{dc}, T_n^{ba} \right] = 0 \quad (4.4)$$

with $T_{-1}^{ab} \equiv \delta_{ab} \mathbf{1}$. The above can be recasted into the more familiar form given by

$$\left[T_n^{ab}, T_m^{cd} \right\} = \delta_{bc} T_{n+m}^{ad} - (-1)^{(p(a)+p(b))(p(c)+p(d))} \delta_{ad} T_{n+m}^{cb} \quad (4.5)$$

Hence the symmetry becomes the $gl(n|m)$ loop algebra. The central elements are simply given by

$$Q_s = \sum_a T_s^{aa}, \quad s \geq 0 \quad (4.6)$$

which for the case $n = m$, satisfy

$$Q_{s+p} = \sum_{a,b}' \left\{ T_s^{ab}, T_p^{ba} \right\} \quad (4.7)$$

where $\{ \, , \, \}$ is the usual anti-commutator and $\sum'$ denotes summation restricted to color indices with $p(a) = 0, p(b) = 1$. These generators have grading 1 and satisfy

$$\left\{ T_s^{ab}, T_p^{cd} \right\} = 0, \quad \text{for } p(a) = p(c) \neq p(b) = p(d)$$

Therefore they can be regarded as supersymmetric charges. In particular, the Hamiltonian, which is simply $\frac{1}{2} Q_2$, is written as

$$H = \frac{1}{2} \sum_{a,b}' \left\{ T_1^{ab}, T_1^{ba} \right\}$$

In this light, we can consider the $Y(gl(n|m))$ Yangian algebra as a deformation of the SUSY algebra given above. Only in the rational limit, the $gl(n|m)$ CS model can have the supersymmetry.

Restricting further to the case where the symmetry is the $gl(1|1)$ loop algebra, the model is the supersymmetric CS model studied in [7] without the harmonic potential. To see this, consider N copies of Grassmanian variables and their conjugates $\{\theta_j, \theta_j^\dagger\}$, they satisfy the usual anti-commutation relations

$$\{\theta_j, \theta_k\} = \{\theta_j^\dagger, \theta_k^\dagger\} = 0, \quad \{\theta_j, \theta_k^\dagger\} = \delta_{jk} \quad (4.8)$$

For a given j, the set $\{\theta_j, \theta_j^\dagger\}$ furnishes a two dimensional representation for the $gl(1|1)$ as

$$\begin{aligned} X_j^{12} &= \theta_j^\dagger & X_j^{21} &= \theta_j \\ X_j^{11} &= \theta_j^\dagger \theta_j & X_j^{22} &= \theta_j \theta_j^\dagger \end{aligned} \quad (4.9)$$

The T_1^{ab} generator given by

$$T_1^{ab} = \sum_j X_j^{ab} \frac{\partial}{\partial x_j} + \lambda' \sum_{j \neq k} \omega_{jk} (X_j^{aa} X_k^{ab} (-1)^{p(a)} + X_j^{ab} X_k^{bb} (-1)^{p(b)}), \quad a \neq b \quad (4.10)$$

can be simplified by noting that the last term can be combined into $\lambda' \omega_{jk}(-1)^{p(a)} X_j^{ab}$. Upon substituting the Grassmanian variables for X_j^{ab}, we get

$$T_1^{12} = \sum_j \theta_j^\dagger \left(\frac{\partial}{\partial x_j} + \lambda' \sum_{k \neq j} \omega_{jk} \right) \qquad (4.11)$$

$$T_1^{21} = \sum_j \theta_j \left(\frac{\partial}{\partial x_j} - \lambda' \sum_{k \neq j} \omega_{jk} \right) \qquad (4.12)$$

which coincide with the SUSY charges $Q^\dagger, Q$ in [7].

Another limit which is of interest is given by $\lambda \rightarrow \infty$, where the L-operator takes the simplified form

$$L_{jk} = \lambda \theta_{jk} P_{jk}$$

Since $T_s^{ab} \sim \lambda^s$, the Yangian relation (1.3) actually does not depend on λ and remains finite in this limit. Thus T_s^{ab}'s constructed with the above L-operator give a representation of the Yangian $Y(gl(n|m))$. However, one can check that with this representation, the generator of the central elements $Z(u)$ is proportional to the identity operator, this can be easily seen in Q_0, Q_1, Q_2 keeping only the most dominant terms. On the other hand, consider the expression for H in (3.13). It can be shown that the second term given by

$$\frac{1}{2} \lambda \sum_{j \neq k} \partial_j \theta_{jk} P_{jk} \qquad (4.13)$$

continues to commute with the Yangian generators. One can then take the above as the Hamiltonian of a spin system, where spins here refer to the elements of the vector module $V^{\otimes N}$ of $gl(n|m)$ and the spins are fixed at equilibrium positions of the CS particles. This is therefore a supersymmetric extension of the Haldane-Shastry spin chain. The motivation for ignoring the most dominant term in H is that it is an unimportant constant, it is not clear whether one can also obtain other commuting conserved quantities from central elements of the origin Yangian $Y(gl(n|m))$ following the above procedure. Nonetheless, one can employ the approach used in [5] to construct other commuting conserved quantities and show that this spin system is integrable.

On the other hand, it is well-known [3] that the $SU(2)$ Haldane-Shastry model corresponds to CS model without color for a special value of λ. In our case, we expect a similar correspondence appears so that $gl(1|1)$ CS model may be related to the spin chain model with $gl(1|2)$ color. Furthermore, it is likely that one may find a relevance of this spin chain model with the supersymmetric $t - J$ model considered in [11].

5. Conclusion

The present paper gives a unified treatment for the color $gl(n|m)$ Calogero-Sutherland models. We prove that the family of models is integrable with the Yangian $Y(gl(n|m))$ as the underlying symmetry. This family of the models include the well studied $gl(n)$ extension of CS model. In the rational limit, we demonstrate how the supersymmetry charges arise based on the $gl(n|m)$ loop algebra.

The work is a modest effort in trying to understand the underlying algebraic structure of the integrable long range interacting models. By identifying the common feature, one may hope to extend the construction to models related to other Lie algebra as in the case of integrable nearest-neighbor interacting spin systems. In fact there are more general CS type models associated with all root systems of the simple Lie algebra [10], it would be interesting to identify the symmetry of these models.

One can also relate the long range nonrelativistic model to the factorizable scattering theory without implementing the crossing symmetry [12]. Here, we claim that the supersymmetric factorizable S-matrix [13] in the nonrelativistic limit can be obtained from the two-particle wave functions of the supersymmetric CS model with $1/\sinh^2(x)$ potential [14].

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