

Deformed type 0A matrix model and super-Liouville theory for fermionic black holes

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ABSTRACT: We consider a $\hat{c} = 1$ model in the fermionic black hole background. For this purpose we consider a model which contains both the $N = 1$ and the $N = 2$ super-Liouville interactions. We propose that this model is dual to a recently proposed type 0A matrix quantum mechanics model with vortex deformations. We support our conjecture by showing that non-perturbative corrections to the free energy computed by both the matrix model and the super-Liouville theories agree exactly by treating the $N = 2$ interaction as a small perturbation. We also show that a two-point function on sphere calculated from the deformed type 0A matrix model is consistent with that of the $N = 2$ super-Liouville theory when the $N = 1$ interaction becomes small. This duality between the matrix model and super-Liouville theories leads to a conjecture for arbitrary n -point correlation functions of the $N = 1$ super-Liouville theory on the sphere.

KEYWORDS: Conformal Field Models in String Theory, Matrix Models.

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1. Introduction

Low dimensional (super-)string theories, minimal conformal field theories (CFTs) coupled with world sheet quantum (super-)gravity [1] have been actively studied mainly because they can be toy models useful to test many interesting conjectures which are difficult to deal with in higher dimensions. Such dualities as holography and open/closed string duality can be quantitatively confirmed by calculating rigorously various amplitudes from both the world sheet formulations and their dual matrix models. Another important application of the (super-)Liouville theories is to investigate black hole physics. One of crucial developments is that 2D euclidean black hole constructed by a coset CFT $SU(2, R)/U(1)$ [2] is dual to the sine-Liouville theory [3].

Dual matrix model for the black hole is provided by Kazakov, Kostov, and Kutasov [4]. These authors have investigated the $c = 1$ matter CFT coupled with the Liouville theory which is perturbed by the sine-Liouville interaction. When the cosmological constant, the coefficient of the Liouville interaction, is much bigger than that of the sine-Liouville interaction, the model is basically the $c = 1$ string theory and described by conventional matrix quantum mechanics. The opposite limit in which the sine-Liouville interaction dominates describes the Witten's 2D euclidean black hole. The corresponding matrix model can be constructed by adding vortex operators to the matrix quantum mechanics.

This conjecture is proved by computing the free energies of both theories directly. The free energy of the deformed matrix model with infinite number of vortex deformations is identified with the τ -function of the Toda chain hierarchy. Allowing only the sine-Liouville perturbation, the hierarchy is simplified to the Toda equation, which can be solved exactly to find the free energy as a function of the cosmological constant and the coefficient of the sine-Liouville interaction. The free energy can be expressed in terms of an infinite sum of correlation functions of the sine-Liouville interaction terms. Using G. Moore's conjecture for the correlation functions [5], one can sum up the infinite terms to derive the free energy which agrees with the matrix model result.

The dual matrix model for the coupled Liouville theory is further confirmed by Alexandrov, Kazakov and Kutasov who computed non-perturbative corrections of both theories [6]. The free energy of a string theory is given by the genus expansion

$$F_{\text{pert}} = \sum_{g=0}^{\infty} g_s^{2g-2} f_g \tag{1.1}$$

where g_s is the string coupling constant and g is the genus of the Riemann surfaces. This perturbative free energy is corrected by non-perturbative effects which are given as a form of

$$F_{\text{non-pert}} \approx C g_s^{f_A} e^{-f_D/g_s}. \tag{1.2}$$

These effects are the exponentiation of the disk partition sum of the D-instanton. To work out the non-perturbative effects in the Liouville theory is a difficult task because the understanding of the D-brane involves the strong coupling effect as $g_s \sim \exp(Q\phi)$ with ϕ representing Liouville direction. This problem was solved by [7, 8] where the extended (FZZT) and localized (ZZ) branes were found. For the Liouville theory, the leading non-perturbative correction is given by the fundamental ZZ-brane and consistent with the matrix model result.

In this paper we generalize these developments to the supersymmetric theories. The matrix model for the fermionic black holes has been constructed by adding vortex deformations to the type 0A matrix quantum mechanics [9]. Because the type 0A matrix model is dual to the $\hat{c} = 1$ conformal matter coupled by the $N = 1$ super-Liouville field theory (SLFT) [10] and the fermionic black hole to the $N = 2$ SLFT [11], it is natural to conjecture that the deformed type 0A matrix model is dual to the $N = 1$ super-Liouville theory perturbed by the $N = 2$ SLFT interaction.

We support our conjecture using many recent developments on the SLFT theories. For the $N = 1$ SLFT, the structure constants [12, 13] and boundary one-point functions [14, 15] have been calculated. Similar results for the $N = 2$ SLFT have been obtained in [16–23]. Using these results, we will calculate the free energy of the coupled SLFT and non-perturbative corrections due to the D-branes and show that they agree with the deformed type 0A matrix model results.

This paper is organized as follows: In section 2 we calculate the free energy and non-perturbative corrections of the deformed type 0A matrix model based on [9]. The $N = 1$ SLFT perturbed by the $N = 2$ SLFT interaction is defined in section 3. To compare with

matrix model results, we compute the non-perturbative corrections based on the boundary one-point functions of the $N = 1$ SLFT and a bulk two-point function of the $N = 2$ SLFT. We conclude the paper with discussions and open questions in section 4. We provide necessary formulae of the $N = 1$ and $N = 2$ SLFTs in the appendices.

2. Deformed type 0A matrix quantum mechanics

2.1 Perturbative expansion of the free energy

In this section, we set $\alpha' = 1/2$. A matrix model of the type 0A string theory in two dimensions without R-R flux is proposed [10]

$$S_{0A} = \int dt \text{Tr} \left[|Dt|^2 - |t|^2 \right]. \tag{2.1}$$

Here t is an $N \times N$ complex matrix and Dt is the covariant derivative. We will consider the case in which the time direction is compactified on S^1 with radius R , so there are two non-trivial Wilson loops $\text{Tr}\Omega$, $\text{Tr}\tilde{\Omega}$ for two gauge fields. As in [4], one can deform the type 0A matrix model (2.1) as follows [9]:

$$S = S_{0A} + \sum_{n \neq 0} \left[\lambda_n \text{Tr} \Omega^n + \tilde{\lambda}_n \text{Tr} \tilde{\Omega}^n \right]. \tag{2.2}$$

When the parameters are given by

$$\lambda_{\pm 1} = \tilde{\lambda}_{\pm 1} = \lambda, \quad \lambda_{n \neq \pm 1} = \tilde{\lambda}_{n \neq \pm 1} = 0, \tag{2.3}$$

this corresponds to a matrix model of Witten's black hole in type 0A string theory.

When $\lambda_n = \tilde{\lambda}_n$, the partition function of the deformed matrix model is factorized as follows,

$$\begin{aligned} Z_{0A}(\lambda, \mu) &= \sum_{N=0}^{\infty} e^{2\pi R \mu N} \int D\Omega D\tilde{\Omega} Dt e^{-S} \\ &= Z_+(\lambda, \mu) \cdot Z_-(\lambda, \mu). \end{aligned} \tag{2.4}$$

Moreover, when the condition (2.3) holds, the free energy of each factor defined by $F_{\pm}(\lambda, \mu) := \log Z_{\pm}(\lambda, \mu)$ satisfies the following non-linear differential equation,

$$\frac{1}{4} \lambda^{-1} \partial_{\lambda} (\lambda \partial_{\lambda} F_{\pm}(\lambda, \mu')) + \exp \left[-\sin^2(\partial_{\mu'} F_{\pm}(\lambda, \mu')) \right] = 1, \tag{2.5}$$

where $\mu' = \mu/2$. From now on, we will drop the prime.

One can solve this equation at least perturbatively, by using the free energy at $\lambda = 0$

$$F_{\pm}(\lambda = 0, \mu) = -\frac{1}{4} \text{Re} \int_0^{\infty} \frac{dt}{t} \frac{e^{-it\mu}}{\sinh(t/2) \sinh(t/4R)} \mp \frac{i}{4} \text{Im} \int_0^{\infty} \frac{dt}{t} \frac{e^{-it\mu}}{\sinh(t/2) \cosh(t/4R)}, \tag{2.6}$$

as a boundary condition. For the tree level free energy $F_{\pm,0}(\lambda, \mu)$ the susceptibility

$$\chi_{\pm,0} = \partial_{\mu}^2 F_{\pm,0} = -\frac{2R}{1-R} \log \lambda + X_{\pm,0}(y), \tag{2.7}$$

satisfies the following differential equation

$$(y\partial_y)^2 X_{\pm,0}(y) + 4(1-R)^2 \partial_y^2 e^{-X_{\pm,0}(y)} = 0, \quad (2.8)$$

where $y = \mu/\lambda^{1/(1-R)}$. The solution is

$$y = e^{-X_{\pm,0}/2R} - (2R-1)e^{(1-2R)X_{\pm,0}/2R}. \quad (2.9)$$

From this we can derive

$$\chi_{\pm,0}(\lambda, \mu) = -2R \log \mu + 2R \log(1-s) \quad (2.10)$$

where

$$\frac{s}{(1-s)^{2-2R}} = (2R-1)\mu^{2R-2}\lambda^2. \quad (2.11)$$

Notice that this result is the same as that of [4] if we rescale R to $2R$.

Eq. (2.10) can be used to expand the susceptibility $\chi_{\pm,0}$. For the large y (small λ), the susceptibility becomes

$$\chi_{\pm,0}(\lambda, \mu) = -2R \log \mu + 2R \sum_{n=1}^{\infty} \frac{1}{n!} [(1-2R)\mu^{2R-2}\lambda^2]^n \frac{\Gamma(n(2-2R))}{\Gamma(n(1-2R)+1)}. \quad (2.12)$$

Integrating this twice we obtain the free energy

$$F_{\pm,0}(\lambda, \mu) = -R\mu^2 \log(\mu/\Lambda) + 2R\mu^2 \sum_{n=1}^{\infty} \frac{1}{n!} [(1-2R)\mu^{2R-2}\lambda^2]^n \frac{\Gamma(n(2-2R)-2)}{\Gamma(n(1-2R)+1)}. \quad (2.13)$$

When λ becomes strong, we can take a series expansion of the $N=1$ SLFT perturbation for the underlying $N=2$ SLFT. From eq. (2.10) we obtain for $1/2 < R < 1$

$$\begin{aligned} \chi_{\pm,0}(\lambda, \mu) = & -\frac{R}{1-R} \log((2R-1)\lambda^2) + \\ & + \frac{R}{1-R} \sum_{n=1}^{\infty} \frac{1}{n!} \left[-\mu ((2R-1)\lambda^2)^{\frac{1}{2-2R}} \right]^n \frac{\Gamma\left(\frac{n}{2-2R}\right)}{\Gamma\left(\frac{n}{2-2R} - n + 1\right)}. \end{aligned} \quad (2.14)$$

When $R=1/2$ which corresponds to the fermionic black hole, one needs to deal with eq. (2.13) carefully because the first two terms becomes singular while the expansion parameter vanishes. A careful calculation leads to [9]

$$F_{\pm,0}(\lambda, \mu) = -\frac{\mu^2}{2} \log \mu - \mu\lambda^2. \quad (2.15)$$

We will compare these results with those of the $N=2$ SLFT in section 3.

2.2 Non-perturbative corrections

Since the full $F_{\pm}(\lambda, \mu)$, not only the perturbative contributions, is a solution of (2.5), one can also derive non-perturbative contributions from the equation. Suppose that the free energy is of the following form,

$$F_{\pm}(\lambda, \mu) = F_{\pm, \text{pert}}(\lambda, \mu) + \epsilon_{\pm}(\lambda, \mu), \quad (2.16)$$

where the first term of the RHS contains all perturbative contributions, and non-perturbative contributions are included in the second term. If we obtained $F_{\pm, \text{pert}}(\lambda, \mu)$, which is possible in principle, one could obtain $\epsilon_{\pm}(\lambda, \mu)$ by solving

$$\frac{1}{4}\lambda^{-1}\partial_{\lambda}(\lambda\partial_{\lambda}\epsilon_{\pm}(\lambda, \mu)) = \exp\left[-4\sin^2\left(\frac{\partial_{\mu}}{2}\right)F_{\pm, \text{pert}}(\lambda, \mu)\right]\left\{1 - \exp\left[-4\sin^2\left(\frac{\partial_{\mu}}{2}\right)\epsilon_{\pm}(\lambda, \mu)\right]\right\}. \quad (2.17)$$

One may notice from (2.6) that the leading non-perturbative term would behave like $e^{-2\pi R\mu}$ coming from the imaginary part of $F_{\pm}(\lambda = 0, \mu)$. This contribution is not physical since this term would be cancelled in the total free energy $F_{0A}(\lambda, \mu) = F_{+}(\lambda, \mu) + F_{-}(\lambda, \mu)$. So the physically relevant term would come from the next-leading terms. We denote the leading non-perturbative term by $\epsilon_0(\lambda, \mu)$ and the next-to-leading term by $\epsilon_1(\lambda, \mu)$. The equations one has to solve are now,

$$\frac{1}{4}\lambda^{-1}\partial_{\lambda}(\lambda\partial_{\lambda}\epsilon_0(\lambda, \mu)) = 4\exp\left[-4\sin^2\left(\frac{\partial_{\mu}}{2}\right)F_{\pm, \text{pert}}(\lambda, \mu)\right]\sin^2\left(\frac{\partial_{\mu}}{2}\right)\epsilon_0(\lambda, \mu), \quad (2.18)$$

$$\begin{aligned} \frac{1}{4}\lambda^{-1}\partial_{\lambda}(\lambda\partial_{\lambda}\epsilon_1(\lambda, \mu)) &= \exp\left[-4\sin^2\left(\frac{\partial_{\mu}}{2}\right)F_{\pm, \text{pert}}(\lambda, \mu)\right] \times \\ &\times \left\{4\sin^2\left(\frac{\partial_{\mu}}{2}\right)\epsilon_1(y, \mu) - 8\left(\sin^2\left(\frac{\partial_{\mu}}{2}\right)\epsilon_0(y, \mu)\right)^2\right\}. \end{aligned} \quad (2.19)$$

In solving (2.18) and (2.19), we use the tree level free energy expression $F_{\pm, 0}(\lambda, \mu)$ for $F_{\pm, \text{pert}}(\lambda, \mu)$. By making the following ansatz for $\epsilon_0(\lambda, \mu)$

$$\epsilon_0(\lambda, \mu) = P_0(y, \mu)e^{-\mu f(y)}, \quad (2.20)$$

one can obtain, from (2.18), the following equation for $f(y)$,

$$\frac{1}{2(1-R)}(1-y\partial_y)g(y) = \pm e^{-X(y)/2}\sin(\partial_y g(y)), \quad (2.21)$$

where $g(y) = yf(y)/2$ and $X(y) \equiv X_{\pm, 0}$. We impose the boundary condition $f(y) \rightarrow 2\pi R$ as $y \rightarrow \infty$. Then we obtain the solution

$$f(y) = 2\pi R \pm 4\sin(\pi R)\lambda\mu^{R-1} + \dots, \quad (2.22)$$

for large μ . Note that R should be less than one in order to obtain $F_{\pm, \text{pert}}$ obeying the boundary condition.

In deriving (2.19), we assumed that $\epsilon_1 \sim (\epsilon_0)^2$. Therefore $\epsilon_1(\lambda, \mu)$ should behave as

$$\epsilon_1(\lambda, \mu) = P_1(y, \mu)e^{-2\mu f(y)}. \quad (2.23)$$

One can show that this assumption is consistent with (2.19). Moreover, one can obtain both $P_0(y, \mu)$ and $P_1(y, \mu)$ perturbatively in terms of $1/\mu$. These $1/\mu$ corrections arise if we use the free energy expressions including contributions from higher genera. It is nontrivial that the relation $\epsilon_1 \sim \epsilon_0^2$, which holds for $\lambda = 0$, persists for nonzero λ since the governing equations for ϵ_0, ϵ_1 are nonlinear.

Now we calculate r_m and ρ_m in the deformed matrix model. The definitions of r, ρ in the matrix model are

$$r_m = \frac{\partial_\mu \log \epsilon_1}{\sqrt{|\partial_\mu^2 (F_{+,0} + F_{-,0})|}} \Big|_{\lambda=0}, \tag{2.24}$$

$$\rho_m = \frac{\partial_\lambda \log \epsilon_1}{\sqrt{|\partial_\lambda^2 (F_{+,0} + F_{-,0})|}} \Big|_{\lambda=0}. \tag{2.25}$$

Corresponding definitions in the Liouville theory are given by

$$r = \frac{\partial_{\mu_L} Z_{disk}}{\sqrt{|\partial_{\mu_L}^2 F_0|}} \Big|_{\lambda=0}, \tag{2.26}$$

$$\rho = \frac{\partial_{\lambda_L} Z_{disk}}{\sqrt{|\partial_{\lambda_L}^2 F_0|}} \Big|_{\lambda_L=0} \tag{2.27}$$

As explained in [6], $\log \epsilon$ is the disk partition sum corresponding to the D-instanton. In the definition of r and ρ , the proportional constants between μ, λ and μ_L, λ_L are canceled. When μ_L and λ_L are defined in the next section, we just write them as μ and λ .

From the above results, one can obtain

$$r_m = -\frac{2\pi\sqrt{R}}{\sqrt{\log(\Lambda/\mu)}}, \tag{2.28}$$

$$\rho_m = -4 \sin(\pi R), \tag{2.29}$$

and therefore,

$$\frac{\rho_m}{r_m} = \frac{2}{\pi} \sqrt{\frac{1}{R} \log \frac{\Lambda}{\mu}} \sin(\pi R). \tag{2.30}$$

In order to compare the computations in the Liouville theory where $\alpha' = 2$ is used we simply replace R by $\frac{R}{2}$. Thus in this convention

$$r_m = -\frac{\pi\sqrt{2R}}{\sqrt{\log(\Lambda/\mu)}}, \tag{2.31}$$

$$\rho_m = -4 \sin\left(\frac{\pi R}{2}\right). \tag{2.32}$$

and

$$\frac{\rho_m}{r_m} = \frac{2}{\pi} \sqrt{\frac{2}{R} \log \frac{\Lambda}{\mu}} \sin\left(\frac{\pi R}{2}\right). \tag{2.33}$$

We will confirm these non-perturbative corrections by using the $N = 1$ SLFT results in section 3.

3. Coupled model of the $N = 1$ and $N = 2$ SLFTs

3.1 $N = 1$ SLFT with the winding perturbation

The $N = 1$ SLFT with additional sine-Liouville type interaction was also considered in [9] as a $N = 1$ generalization of the dual relation between the coset CFT $SU(2, R)/U(1)$ of the euclidean 2D black hole and the Liouville theory with the sine-Liouville interaction. It is natural to conjecture that the type 0A matrix model considered in the previous section is dual to the following world sheet action:

$$S = \int d^2z d^2\theta \left[\frac{1}{2\pi} (DX\bar{D}X + D\Phi\bar{D}\Phi) + i\mu e^{b\Phi} + (i\lambda e^{i\frac{R}{2}\tilde{X} + (1-\frac{R}{2})\Phi} + \text{c.c.}) \right], \quad (3.1)$$

where *c.c.* denote complex conjugate while X , $\tilde{X} = X_L - X_R$ and Φ are real scalar super-fields

$$\begin{aligned} X &= x + i\theta\xi + i\bar{\theta}\bar{\xi} + i\theta\bar{\theta}G, \\ \tilde{X} &= \tilde{x} + i\theta\xi - i\bar{\theta}\bar{\xi}, \quad \tilde{x} = x_L - x_R, \\ \Phi &= \phi + i\theta\psi + i\bar{\theta}\bar{\psi} + i\theta\bar{\theta}F. \end{aligned}$$

In terms of the component fields this action can be written as

$$S = \int d^2z (\mathcal{L}_I + \mathcal{L}_{II}), \quad (3.2)$$

$$\mathcal{L}_I = \frac{1}{2\pi} (\partial x \bar{\partial} x + \xi \bar{\partial} \xi + \bar{\xi} \partial \bar{\xi} + \partial \phi \bar{\partial} \phi + \psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}) + i\mu \psi \bar{\psi} e^{b\phi}, \quad (3.3)$$

$$\begin{aligned} \mathcal{L}_{II} &= i\lambda [c_1 \xi \bar{\xi} - ic_2 (\psi \bar{\xi} - \xi \bar{\psi}) + c_3 \psi \bar{\psi}] e^{i\frac{R}{2}\tilde{x} + (1-\frac{R}{2})\phi} + \text{c.c.} + \\ &+ \frac{\pi}{2} (: \mu e^{b\phi} + \lambda \left(1 - \frac{R}{2}\right) e^{i\frac{R}{2}\tilde{x} + (1-\frac{R}{2})\phi} + \text{c.c.} :)^2 \end{aligned} \quad (3.4)$$

with

$$c_1 = \frac{R^2}{4}, \quad c_2 = \frac{R}{2} \left(1 - \frac{R}{2}\right), \quad c_3 = \left(1 - \frac{R}{2}\right)^2. \quad (3.5)$$

Here we have followed conventions in [19] where

$$\begin{aligned} \alpha' &= 2, & \partial &= (\partial_x - i\partial_y)/2, & \bar{\partial} &= (\partial_x + i\partial_y)/2, \\ D &= \frac{\partial}{\partial\theta} + \theta\partial, & \bar{D} &= \frac{\partial}{\partial\bar{\theta}} + \bar{\theta}\bar{\partial}, & \int d^2z d^2\theta &= \int dx dy d\bar{\theta} d\theta. \end{aligned} \quad (3.6)$$

As pointed out before, we need to rescale $R \rightarrow R/2$ for $\alpha' = 2$ convention. So the fermionic black hole described by the matrix model with $R = 1/2$ corresponds to $R = 1$ in the SLFT.

When we set $\lambda = 0$, the theory is the $N = 1$ SLFT coupled with $\hat{c} = 1$ free CFT where the background charge term $Q\partial^2\phi$ is given by $Q = b + \frac{1}{b}$. The central charge of the $N = 1$ SLFT is

$$\hat{c}_L = 1 + 2Q^2. \quad (3.7)$$

To cancel the conformal anomaly, we should set $b = 1$ or $Q = 2$ so that total central charge satisfies $\hat{c}_m + \hat{c}_L = 10$.

The bulk (NS) primary fields of the $N = 1$ SLFT are given by

$$V_\alpha(z, \bar{z}) = e^{\alpha\phi} \tag{3.8}$$

with conformal dimension

$$\Delta_\alpha = \frac{1}{2}\alpha(Q - \alpha). \tag{3.9}$$

The last term in eq. (3.4) is a contact term which can be neglected since it does not contribute in CFT calculation. It is convenient to define a dimensionless parameter

$$z = \frac{\lambda}{\mu^{1-\frac{R}{2}}}. \tag{3.10}$$

When $z \ll 1$, \mathcal{L}_{II} can be considered as a marginal perturbation for $0 < R < 2$. In this region we can compute the effect of the \mathcal{L}_{II} interaction on the $N = 1$ SLFT D-branes perturbatively. Another interesting region is $z \gg 1$ where we should treat the last term in \mathcal{L}_I which is the $N = 1$ SLFT interaction as a perturbation. In particular the \mathcal{L}_{II} becomes the $N = 2$ SLFT interaction when $R = 1$ since it can be written as

$$\mathcal{L}_{II} = \frac{i\lambda}{2}\psi^-\bar{\psi}^-e^{\hat{b}\phi^+} + c.c., \tag{3.11}$$

by redefining the fields

$$\phi^\pm = \frac{1}{\sqrt{2}}(\phi \pm i\tilde{x}), \quad \psi^- = \frac{1}{\sqrt{2}}(\psi + i\xi), \quad \bar{\psi}^- = \frac{1}{\sqrt{2}}(\bar{\psi} - i\bar{\xi}). \tag{3.12}$$

Here \hat{b} , the $N = 2$ SLFT coupling constant, becomes $\hat{b} = 1/\sqrt{2}$. It is easy to check that the $Q\partial^2\phi$ term with $Q = 2$ now acts as the background charge of the $N = 2$ SLFT. This means that $z \gg 1$ limit of the total theory can be interpreted as $N = 2$ SLFT perturbed by the $N = 1$ SLFT interaction. In particular the action with $\mu = 0$ gives the $N = 2$ SLFT which describes the fermionic black hole, the super $SU(2, R)/U(1)$ coset CFT.

3.2 Non-perturbative corrections

Now we compute the non-perturbative corrections in the small λ or small z limit. From eq. (2.26), the parameter r is given by

$$r = \frac{\partial_\mu Z_{\text{disk}}}{\sqrt{-\partial_\mu^2 F_0}} = \frac{iZ_{\hat{c}=1}\langle\psi\bar{\psi}e^\phi\rangle_{\text{disk}}}{\sqrt{\langle\psi\bar{\psi}e^\phi\psi\bar{\psi}e^\phi\rangle_{\text{sphere}}}}. \tag{3.13}$$

Here $Z_{\hat{c}=1}$ is the partition function of the $\hat{c} = 1$ matter CFT on the disk with Neumann boundary condition.

The parameter r is given by a disk one-point function and a bulk two-point function. These correlation functions have been computed using both conformal and modular bootstrap methods in [12–15]. Since the cosmological constant operator is a superconformal descendant state of $e^{b\phi}$ with $b = 1$, the correlation functions of this operator are related to those of $e^{b\phi}$ by [14]

$$\langle\psi\bar{\psi}e^{b\phi}\rangle = i\eta\frac{1}{b^2}\langle e^{b\phi}\rangle, \quad \langle\psi\bar{\psi}e^{b\phi}\psi\bar{\psi}e^{b\phi}\rangle = -\frac{1}{b^4}\langle e^{b\phi}e^{b\phi}\rangle. \tag{3.14}$$

Here $\eta = \pm 1$ is a discrete parameter in the boundary conditions on the fermion, $\psi + i\eta\bar{\psi} = 0$. The one-point function $\langle e^{\alpha\phi} \rangle$ is proportional to the boundary wave function for the ZZ-brane with (1, 1) boundary condition given by

$$\Psi_{(1,1)}^{NS}(\alpha) = \pi \left[\mu\pi\gamma \left(\frac{bQ}{2} \right) \right]^{\frac{1}{b}(\frac{Q}{2}-\alpha)} \left[\left(\alpha - \frac{Q}{2} \right) \Gamma(b(\frac{Q}{2} - \alpha)) \Gamma(\frac{1}{b}(\frac{Q}{2} - \alpha)) \right]^{-1}. \quad (3.15)$$

Therefore the one-point function with $\alpha = b$ is given by

$$\langle e^{b\phi} \rangle = C\Psi_{(1,1)}^{NS}(b) = -C\pi \left[\mu\pi\gamma \left(\frac{bQ}{2} \right) \right]^{\frac{1}{2}(\frac{1}{b^2}-1)} \left[b\Gamma(\frac{1}{2}(1-b^2))\Gamma(\frac{1}{2}(\frac{1}{b^2}+1)) \right]^{-1} \quad (3.16)$$

where C is a proportional constant between the one-point function and boundary wave function.

The two-point function on the sphere $\langle e^{b\phi} e^{b\phi} \rangle$ has been computed in appendix A and the result is

$$\langle e^{b\phi} e^{b\phi} \rangle = \frac{b^{-1}}{4\pi} \left(\frac{1}{b^2} - 1 \right) \left[\mu\pi\gamma \left(\frac{bQ}{2} \right) \right]^{\frac{1}{b^2}-1} \gamma \left(\frac{bQ}{2} \right) \gamma \left(\frac{1}{2} \left(1 - \frac{1}{b^2} \right) \right). \quad (3.17)$$

As $b \rightarrow 1$, the parameter r can be computed from eqs. (3.14), (3.16), and (3.17) as follows:

$$r = -\lim_{b \rightarrow 1} \frac{Z_{\hat{c}=1} \eta \langle e^{b\phi} \rangle}{\sqrt{-\langle e^{b\phi} e^{b\phi} \rangle}}, \quad (3.18)$$

$$\langle e^{b\phi} \rangle = -C\pi \left[\mu\pi\gamma \left(\frac{bQ}{2} \right) \right]^{\frac{1}{2}(\frac{1}{b^2}-1)} \frac{1}{\Gamma(\frac{1}{2}(1-b^2))}, \quad (3.19)$$

$$\langle e^{b\phi} e^{b\phi} \rangle = -\frac{1}{2\pi} \left[\mu\pi\gamma \left(\frac{bQ}{2} \right) \right]^{\frac{1}{b^2}-1} \frac{1}{\Gamma(\frac{1}{2}(1-b^2))}. \quad (3.20)$$

To compute $\hat{c} = 1$ partition function $Z_{\hat{c}=1}$, it is helpful to split this into $c = 1$ free boson and $c = 1/2$ fermion. The free boson part with Neumann boundary condition is given by $Z_{c=1} = \sqrt{R/2}$ [6]. The $c = 1/2$ free fermion system is equivalent to a minimal CFT for the critical Ising model. Its disk partition function $Z_{c=1/2}$ is the $p = 3$ case of

$$Z_{r,s} = \left(\frac{8}{p(p+1)} \right)^{\frac{1}{4}} \frac{\sin \frac{\pi r}{p} \sin \frac{\pi s}{p+1}}{(\sin \frac{\pi}{p} \sin \frac{\pi}{p+1})^{\frac{1}{2}}}, \quad (3.21)$$

where (r, s) denotes a conformal boundary condition [24]. In this case we choose the Ising model boundary condition $(r, s) = (1, 1)$, which gives $Z_{c=1/2} = Z_{1,1} = 1/\sqrt{2}$. Substituting these into eq. (3.18), we obtain

$$r = C\eta \frac{\pi^{\frac{3}{2}} \sqrt{R}}{\sqrt{2 \log \frac{\Lambda}{\mu}}}. \quad (3.22)$$

This result agrees with the matrix model result (2.31) if we can fix the proportional constant to $C\eta = -\frac{2}{\sqrt{\pi}}$. Since the proportional constant C can be fixed only when it is compared with the matrix model calculation, it is interesting to check if the same C can be obtained

for different matrix models. We will comment on this in the conclusion but for now we just fix the value as it is.

Now let us move on to the next order in λ . We rewrite the $N = 2$ perturbation term (3.4) as

$$\mathcal{T} = \frac{\mathcal{L}_{II}}{2i\lambda} = [c_1 \xi \bar{\xi} - ic_2(\psi \bar{\xi} - \xi \bar{\psi}) + c_3 \psi \bar{\psi}] \cos\left(\frac{R}{2}\tilde{x}\right) e^{(1-\frac{R}{2})\phi} \quad (3.23)$$

where c_i 's are defined in eq. (3.5). From eq. (2.27) the parameter ρ is given by

$$\rho = \frac{\partial_\lambda Z_{\text{disk}}}{\sqrt{-\partial_\lambda^2 F_0}} \Big|_{\lambda=0} = \frac{i\langle \mathcal{T} \rangle_{\text{disk}}}{\sqrt{\langle \mathcal{T}\mathcal{T} \rangle_{\text{sphere}}}}. \quad (3.24)$$

The one-point function $\langle \mathcal{T} \rangle$ in the numerator can be computed as

$$\langle \mathcal{T} \rangle = \left\langle \cos\left(\frac{R}{2}\tilde{x}\right) \right\rangle \left(c_1 \langle \xi \bar{\xi} \rangle \langle V_{1-\frac{R}{2}} \rangle + c_3 \langle \psi \bar{\psi} V_{1-\frac{R}{2}} \rangle \right). \quad (3.25)$$

Similarly the two-point function $\langle \mathcal{T}\mathcal{T} \rangle$ in the denominator can be written as

$$\begin{aligned} \langle \mathcal{T}\mathcal{T} \rangle = \left\langle \cos^2\left(\frac{R}{2}\tilde{x}\right) \right\rangle & \left(-c_1^2 \langle \xi \xi \rangle \langle \bar{\xi} \bar{\xi} \rangle \langle V_{1-\frac{R}{2}} V_{1-\frac{R}{2}} \rangle + c_2^2 \langle \bar{\xi} \bar{\xi} \rangle \langle \psi V_{1-\frac{R}{2}} \psi V_{1-\frac{R}{2}} \rangle + \right. \\ & \left. + c_2^2 \langle \xi \xi \rangle \langle \bar{\psi} V_{1-\frac{R}{2}} \bar{\psi} V_{1-\frac{R}{2}} \rangle + c_3^2 \langle \psi \bar{\psi} V_{1-\frac{R}{2}} \psi \bar{\psi} V_{1-\frac{R}{2}} \rangle \right). \quad (3.26) \end{aligned}$$

The one-point function $\langle \xi \bar{\xi} \rangle$ in eq. (3.25) corresponds to that of the energy operator ϵ in the Ising model: $\langle \epsilon \rangle = i\langle \xi \bar{\xi} \rangle$. The boundary states of the Ising model are well-known and the one-point function $\langle \epsilon \rangle$ is η . The correlation functions of the descendant fields are related to those of the primary field by

$$\begin{aligned} \langle \psi \bar{\psi} V_{1-\frac{R}{2}} \rangle &= i\eta \frac{1+R/2}{1-R/2} \langle V_{1-\frac{R}{2}} \rangle, \\ \langle \psi V_{1-\frac{R}{2}} \psi V_{1-\frac{R}{2}} \rangle &= \frac{1+R/2}{1-R/2} \langle V_{1-\frac{R}{2}} V_{1-\frac{R}{2}} \rangle, \\ \langle \bar{\psi} V_{1-\frac{R}{2}} \bar{\psi} V_{1-\frac{R}{2}} \rangle &= \frac{1+R/2}{1-R/2} \langle V_{1-\frac{R}{2}} V_{1-\frac{R}{2}} \rangle, \\ \langle \psi \bar{\psi} V_{1-\frac{R}{2}} \psi \bar{\psi} V_{1-\frac{R}{2}} \rangle &= -\left(\frac{1+R/2}{1-R/2}\right)^2 \langle V_{1-\frac{R}{2}} V_{1-\frac{R}{2}} \rangle. \end{aligned}$$

Using these relations, ρ can be rewritten as

$$\rho = -\sqrt{2}B\eta \frac{\langle V_{1-\frac{R}{2}} \rangle}{\sqrt{-\langle V_{1-\frac{R}{2}} V_{1-\frac{R}{2}} \rangle}}, \quad (3.27)$$

where B is the one point function of $\cos(\frac{R}{2}\tilde{x})$ on the disk with Neumann boundary conditions, which is known to be the same as $Z_{c=1}$. The one-point function $\langle V_{b-\frac{R}{2}} \rangle$ can be derived from eq. (3.15)

$$\langle V_{b-\frac{R}{2}} \rangle = -C\pi \left[\mu\pi\gamma \left(\frac{bQ}{2}\right) \right]^{\frac{1}{2}(\frac{1}{b^2}-1)+\frac{R}{2b}} \left[b\Gamma\left(\frac{1}{2}(1-b^2) + \frac{bR}{2}\right) \Gamma\left(\frac{1}{2}\left(\frac{1}{b^2}+1\right) + \frac{R}{2b}\right) \right]^{-1}. \quad (3.28)$$

The two-point function $\langle V_{b-\frac{R}{2}} V_{b-\frac{R}{2}} \rangle$ can be computed similarly as before and its derivation is explained in details in appendix A. The result is

$$\begin{aligned} \langle V_{b-\frac{R}{2}} V_{b-\frac{R}{2}} \rangle &= \frac{b^{-1}}{4\pi} \left(\frac{1}{b^2} - 1 + \frac{R}{b} \right) \left[\mu\pi\gamma \left(\frac{bQ}{2} \right) \right]^{\frac{1}{b^2}-1+\frac{R}{b}} \gamma \left(\frac{1}{2}(1+b^2-bR) \right) \times \\ &\times \gamma \left(\frac{1}{2} \left(1 - \frac{1}{b^2} - \frac{R}{b} \right) \right). \end{aligned} \quad (3.29)$$

In the limit $b \rightarrow 1$, the one- and two-point functions approach

$$\langle V_{b-\frac{R}{2}} \rangle = -C\pi \left[\mu\pi\gamma \left(\frac{bQ}{2} \right) \right]^{\frac{1}{2}(\frac{1}{b^2}-1)+\frac{R}{2b}} \frac{1}{\Gamma(R/2)\Gamma(1+R/2)}, \quad (3.30)$$

$$\langle V_{b-\frac{R}{2}} V_{b-\frac{R}{2}} \rangle = -\frac{1}{\pi R} \left[\mu\pi\gamma \left(\frac{bQ}{2} \right) \right]^{\frac{1}{b^2}-1+\frac{R}{b}} \left(\frac{\Gamma(1-R/2)}{\Gamma(R/2)} \right)^2. \quad (3.31)$$

Substituting these into (3.27), we obtain

$$\rho = \pi^{3/2} C\eta \frac{R}{\Gamma(1+R/2)\Gamma(1-R/2)} = 2C\eta\sqrt{\pi} \sin \frac{\pi R}{2}, \quad (3.32)$$

and the ratio

$$\frac{\rho}{r} = \frac{2}{\pi} \sqrt{\frac{2}{R} \log \frac{\Lambda}{\mu}} \sin \frac{\pi R}{2}. \quad (3.33)$$

which confirms the matrix model result eq. (2.33).

3.3 Two-point functions of the $N = 2$ SLFT

An interesting check of our conjecture is to compute the free energy in the large λ limit, eq. (2.15), where the underlying theory is the $N = 2$ SLFT. In this limit, the free energy can be expanded as

$$F_0(\lambda, \mu) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \left\langle \left[i \int d^2z d^2\theta e^{\Phi} \right]^n \right\rangle_{N=2}. \quad (3.34)$$

Here $\langle \dots \rangle_{N=2}$ means that the correlation function is evaluated in the context of the $N = 2$ SLFT.

It is very difficult to derive general n -point correlation functions. But three-point correlation functions for some special operators are computable as we have listed in appendix B. We need the following three-point functions of the $N = 2$ SLFT:

$$\begin{aligned} \langle e^{\hat{b}(\phi^++\phi^-)} e^{\hat{b}(\phi^++\phi^-)} \psi^- \bar{\psi}^- e^{\hat{b}\phi^+} \rangle &= \langle e^{\hat{b}(\phi^++\phi^-)} e^{\hat{b}(\phi^++\phi^-)} \psi^+ \bar{\psi}^+ e^{\hat{b}\phi^-} \rangle \\ &= \lambda^{\frac{2}{\hat{b}^2}-5} \frac{\hat{b}^{-13} \gamma(1-\hat{b}^2)^2 \gamma(2-1/\hat{b}^2) (2\hat{b}^2-1)^2}{\sqrt{2} \gamma(1-2\hat{b}^2)}. \end{aligned} \quad (3.35)$$

Integrating these and using the relation (B.5), we obtain

$$\langle \psi \bar{\psi} e^{\hat{b}(\phi^++\phi^-)} \psi \bar{\psi} e^{\hat{b}(\phi^++\phi^-)} \rangle = -\frac{\sqrt{2}}{2} i \hat{b}^{-9} \left(\frac{1}{\hat{b}^2} - 1 \right)^2 (2\hat{b}^2 - 1) \frac{\gamma(1-\hat{b}^2)^2 \gamma(2-1/\hat{b}^2)}{\gamma(1-2\hat{b}^2)} \lambda^{\frac{2}{\hat{b}^2}-4}. \quad (3.36)$$

When we fix $\hat{b} = 1/\sqrt{2}$, the two-point function vanishes.

Noticing that $\psi\bar{\psi}e^\phi$ is the $N = 1$ SLFT interaction, we can expect that the small μ corrections to the $N = 2$ SLFT free energy should not have μ^2 terms which can be checked easily in the matrix model result (2.15). In fact, this result shows that general n -point ($n \geq 2$) functions of the $N = 1$ SLFT interaction terms vanish. But confirming this in the $N = 2$ SLFT context is beyond the scope of this paper.

4. Conclusion

In this paper we have proposed a world sheet action corresponding to the deformed type 0A matrix model which describes 2D fermionic black hole. The model is the $N = 1$ SLFT perturbed by the $N = 2$ SLFT interaction. To justify our conjecture, we have computed the non-perturbative effects arising from the D-instanton for the $N = 1$ SLFT with $N = 2$ perturbation and its proposed dual matrix model and have shown the agreement between the two results. This confirms our conjecture for the $N = 1$ SLFT side. When the $N = 2$ SLFT interaction becomes strong, we have calculated the correlation function of two $N = 1$ SLFT interaction operators and confirmed that both the $N = 2$ SLFT and the deformed matrix model agree. Although these calculations are not mathematical proof of the duality conjecture, we believe that they are enough to justify the validity.

For non-perturbative corrections, we have computed the r parameter which measures the D-instanton effect of the $N = 1$ SLFT without the $N = 2$ SLFT interaction and the ρ parameter which is the first order $N = 2$ SLFT perturbation effect. Since the dual relation between the $N = 1$ SLFT and the type 0A matrix model is well established [10], we certainly expect that r should match at both sides. Assuming this, our computation shows that ρ matches at both sides, which confirms the same D-instanton effect in the perturbed case. In order to decide the precise value of r we should fix the undetermined quantity $C\eta$ in the computation. This cannot be done solely in the $N = 1$ SLFT setup. Following the approach of [6], one can consider the $N = 1$ SLFT coupled to superminimal models and decide the similar constant $C\eta$ by comparing with the corresponding matrix models and see if we can get the same $C\eta$ for various superminimal models and the $\hat{c} = 1$ model. Our preliminary analysis shows a discrepancy by a factor 2 which we can not explain yet. We may need a better understanding of the string equation governing superminimal models and to work out the non-perturbative effects of such theories as in [25]. This in itself is an interesting topic and we hope we can report the progress elsewhere but is beyond the scope of the Liouville theory calculation, which is of our main interest in this paper.

If this duality is accepted, one can compare the free energy of the deformed matrix model (2.13) with the small λ expansion of the SLFT free energy. This leads to a supersymmetric version of G. Moore's conjecture for the general n -point correlation functions on the sphere of the certain $N = 1$ SLFT operators:

$$\left\langle \mathcal{T}_R^n (\mathcal{T}_R^*)^n \right\rangle_0 = 2Rn! \mu^2 [(1 - 2R)\mu^{R-2}]^n \frac{\Gamma(n(2 - 2R) - 2)}{\Gamma(n(1 - 2R) + 1)} \quad (4.1)$$

where

$$\mathcal{T}_R = i \int d^2z d^2\theta e^{i\frac{R}{2}\tilde{X} + (1-\frac{R}{2})\Phi}. \quad (4.2)$$

Due to technical difficulties of the $N = 1$ SLFT, this conjecture can not be proved by direct calculations. But this result seems natural because the ordinary Liouville theory and the $N = 1$ SLFT share many physical properties like coupling constant duality in common.¹

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A. Two-point functions of the $N = 1$ SLFT

The three-point function of the $N = 1$ SLFT is given by [14]

$$\begin{aligned} -\alpha_1^2 \langle \psi \bar{\psi} N_{\alpha_1} N_{\alpha_2} N_{\alpha_3} \rangle &= \quad (A.1) \\ &= i \left[\mu \pi \gamma \left(\frac{bQ}{2} \right) b^{1-b^2} \right]^{\frac{Q-\sum \alpha_i}{b}} \frac{2\Upsilon'_{NS}(0) \Upsilon_{NS}(2\alpha_1) \Upsilon_{NS}(2\alpha_2) \Upsilon_{NS}(2\alpha_3)}{\Upsilon_R(\alpha_1+2+3-Q) \Upsilon_R(\alpha_1+2-3) \Upsilon_R(\alpha_2+3-1) \Upsilon_R(\alpha_3+1-2)}, \end{aligned}$$

where

$$\Upsilon_{NS}(x) = \Upsilon\left(\frac{x}{2}\right) \Upsilon\left(\frac{x+Q}{2}\right), \quad \Upsilon_R(x) = \Upsilon\left(\frac{x+b}{2}\right) \Upsilon\left(\frac{x+b^{-1}}{2}\right), \quad (A.2)$$

and $\Upsilon(x)$ satisfies the following formulae

$$\begin{aligned} \Upsilon'(0) &= \Upsilon(b), \\ \Upsilon(x+b) &= \gamma(bx) b^{1-2bx} \Upsilon(x), \\ \Upsilon(x+1/b) &= \gamma(x/b) b^{2x/b-1} \Upsilon(x). \end{aligned}$$

For $\alpha_1 = \alpha_2 = \alpha_3 = b$, using the formulae

$$\begin{aligned} \Upsilon'_{NS}(0) &= \frac{1}{2} \Upsilon_R(b), \\ \frac{\Upsilon_{NS}(2b)}{\Upsilon_R(b)} &= \gamma\left(\frac{bQ}{2}\right) b^{1-bQ}, \\ \frac{\Upsilon_{NS}(2b)}{\Upsilon_R(2b-\frac{1}{b})} &= -\gamma\left(\frac{1}{2}\left(1-\frac{1}{b^2}\right)\right) \left(\frac{bQ}{2}-1\right)^2 b^{-2-\frac{1}{b^2}}, \end{aligned}$$

we find

$$-b^2 \langle \psi \bar{\psi} N_b N_b N_b \rangle = -i(\mu\pi)^{\frac{1}{b^2}-2} \left[\gamma\left(\frac{bQ}{2}\right) \right]^{\frac{1}{b^2}} b^{-5} \gamma\left(\frac{1}{2}\left(1-\frac{1}{b^2}\right)\right) \left(\frac{bQ}{2}-1\right)^2. \quad (A.3)$$

¹After the paper was accepted, we were aware of the work [26] which discusses this question in the matrix model side.

Integrating this, we obtain

$$\langle e^{b\phi} e^{b\phi} \rangle = \frac{b^{-1}}{4\pi} \left(\frac{1}{b^2} - 1 \right) \left[\mu\pi\gamma \left(\frac{bQ}{2} \right) \right]^{\frac{1}{b^2}-1} \gamma \left(\frac{bQ}{2} \right) \gamma \left(\frac{1}{2} \left(1 - \frac{1}{b^2} \right) \right). \quad (\text{A.4})$$

For the two-point function $\langle V_{b-\frac{R}{2}} V_{b-\frac{R}{2}} \rangle$, one can integrate a three-point function $\langle \psi\bar{\psi} V_b V_{b-\frac{R}{2}} V_{b-\frac{R}{2}} \rangle$. Setting $\alpha_1 = b$, $\alpha_2 = \alpha_3 = b - \frac{R}{2}$ in (A.2) and using the formulae

$$\begin{aligned} \frac{\Upsilon_{NS}(2b-R)}{\Upsilon_R(b-R)} &= \gamma \left(b \left(\frac{Q}{2} - \frac{R}{2} \right) \right) b^{1+bR-bQ}, \\ \frac{\Upsilon_{NS}(2b-R)}{\Upsilon_R(2b-\frac{1}{b}-R)} &= -\gamma \left(\frac{1}{2} \left(1 - \frac{1}{b^2} - \frac{R}{b} \right) \right) \left(b \left(\frac{Q}{2} - \frac{R}{2} \right) - 1 \right)^2 b^{-2-\frac{1}{b^2}-\frac{R}{b}}, \end{aligned}$$

we find

$$\begin{aligned} -b^2 \langle \psi\bar{\psi} V_b V_{b-\frac{R}{2}} V_{b-\frac{R}{2}} \rangle &= -i \left[\mu\pi\gamma \left(\frac{bQ}{2} \right) \right]^{\frac{1}{b^2}-2+\frac{R}{b}} \gamma \left(\frac{bQ}{2} \right) b^{-5} \gamma \left(b \left(\frac{Q}{2} - \frac{R}{2} \right) \right) \times \\ &\quad \times \gamma \left(\frac{1}{2} \left(1 - \frac{1}{b^2} - \frac{R}{b} \right) \right) \left(b \left(\frac{Q}{2} - \frac{R}{2} \right) - 1 \right)^2. \end{aligned} \quad (\text{A.5})$$

Integrating this, we obtain

$$\begin{aligned} \langle V_{b-\frac{R}{2}} V_{b-\frac{R}{2}} \rangle &= \frac{b^{-1}}{4\pi} \left(\frac{1}{b^2} - 1 + \frac{R}{b} \right) \left[\mu\pi\gamma \left(\frac{bQ}{2} \right) \right]^{\frac{1}{b^2}-1+\frac{R}{b}} \gamma \left(\frac{1}{2} (1 + b^2 - bR) \right) \times \\ &\quad \times \gamma \left(\frac{1}{2} \left(1 - \frac{1}{b^2} - \frac{R}{b} \right) \right). \end{aligned} \quad (\text{A.6})$$

B. Two-point functions of the $N = 2$ SLFT

We consider the vertex operators of the following form

$$V_{m,\bar{m}}^{j(s,\bar{s})} = \exp \left[\sqrt{\frac{2}{k}} (i(m+s)x_L + i(\bar{m}+\bar{s})x_R - j\phi) + isH_L + i\bar{s}H_R \right], \quad (\text{B.1})$$

where $H_{L,R}$ are bosonized fermions and $s, \bar{s} \in Z$ for the NS sector, $s, \bar{s} \in Z + \frac{1}{2}$ for the R sector. The constant k is related to the $N = 2$ SLFT coupling constant \hat{b} by $k = \hat{b}^2$. The dimensions are given by

$$\Delta = -\frac{j(j+1)}{k} + \frac{(m+s)^2}{k} + \frac{s^2}{2}, \quad \bar{\Delta} = -\frac{j(j+1)}{k} + \frac{(\bar{m}+\bar{s})^2}{k} + \frac{\bar{s}^2}{2}, \quad (\text{B.2})$$

and the U(1) charges are given by

$$\omega = \frac{2(m+s)}{k} + s, \quad \bar{\omega} = \frac{2(\bar{m}+\bar{s})}{k} + \bar{s}. \quad (\text{B.3})$$

When $s = \bar{s} = 0$, m and \bar{m} are given by

$$m = \frac{n+wk}{2}, \quad \bar{m} = \frac{n-wk}{2}, \quad (\text{B.4})$$

where n and w are momentum and winding numbers respectively.

The two-point function of the operator $\psi\bar{\psi}e^{\sqrt{2}\hat{b}\phi} = \psi\bar{\psi}e^{\hat{b}(\phi^++\phi^-)}$ satisfies the following relation

$$\langle \psi\bar{\psi}e^{\hat{b}(\phi^++\phi^-)}\psi\bar{\psi}e^{\hat{b}(\phi^++\phi^-)} \rangle = - \left(\frac{1}{\hat{b}^2} - 1 \right)^2 \langle e^{\hat{b}(\phi^++\phi^-)}e^{\hat{b}(\phi^++\phi^-)} \rangle. \quad (\text{B.5})$$

The operator $e^{\hat{b}(\phi^++\phi^-)}$ is identified with the vertex operator of the form (B.1): $e^{\hat{b}(\phi^++\phi^-)} = V_{0,0}^{-k(0,0)}$. The two-point function $\langle e^{\hat{b}(\phi^++\phi^-)}e^{\hat{b}(\phi^++\phi^-)} \rangle$ is computed by integrating the three-point function $\langle e^{\hat{b}(\phi^++\phi^-)}e^{\hat{b}(\phi^++\phi^-)}\psi^-\bar{\psi}^-e^{\hat{b}\phi^+} \rangle$ and $\langle e^{\hat{b}(\phi^++\phi^-)}e^{\hat{b}(\phi^++\phi^-)}\psi^+\bar{\psi}^+e^{\hat{b}\phi^-} \rangle$ with respect to $-i\hat{b}^2\lambda$. The three-point function of $N = 2$ SLFT is given in [23]. Using the result there, we find

$$\langle e^{\hat{b}(\phi^++\phi^-)}e^{\hat{b}(\phi^++\phi^-)}\psi^-\bar{\psi}^-e^{\hat{b}\phi^+} \rangle = \langle V_{0,0}^{-k(0,0)}V_{0,0}^{-k(0,0)}V_{k/2+1,k/2+1}^{-k/2(-1,-1)} \rangle = F_-D_-. \quad (\text{B.6})$$

The expressions of F_- and D_- are written in [23] and they are calculated as

$$\begin{aligned} F_- &= -\frac{1}{\hat{b}^4}\gamma(1-\hat{b}^2)^3, \\ D_- &= \nu^{-\frac{5}{2}\hat{b}^2+1}\frac{\hat{b}^{6\hat{b}^2+2/\hat{b}^2-6}}{\sqrt{2}}\frac{\Gamma(-2\hat{b}+1/\hat{b})^2\Gamma(-\hat{b}+1/\hat{b})}{\Gamma(3\hat{b}-1/\hat{b})\Gamma(\hat{b})\Gamma(2\hat{b})} \\ &= -\lambda^{\frac{2}{\hat{b}^2}-5}\frac{\hat{b}^{-9}}{\sqrt{2}}\frac{\gamma(2-1/\hat{b}^2)(2\hat{b}^2-1)^2}{\gamma(1-2\hat{b}^2)\gamma(1-\hat{b}^2)}, \end{aligned}$$

and we obtain

$$\langle e^{\hat{b}(\phi^++\phi^-)}e^{\hat{b}(\phi^++\phi^-)}\psi^-\bar{\psi}^-e^{\hat{b}\phi^+} \rangle = \lambda^{\frac{2}{\hat{b}^2}-5}\frac{\hat{b}^{-13}}{\sqrt{2}}\frac{\gamma(1-\hat{b}^2)^2\gamma(2-1/\hat{b}^2)(2\hat{b}^2-1)^2}{\gamma(1-2\hat{b}^2)}. \quad (\text{B.7})$$

Similarly we find

$$\begin{aligned} \langle e^{\hat{b}(\phi^++\phi^-)}e^{\hat{b}(\phi^++\phi^-)}\psi^+\bar{\psi}^+e^{\hat{b}\phi^-} \rangle &= \langle V_{0,0}^{-k(0,0)}V_{0,0}^{-k(0,0)}V_{-k/2-1,-k/2-1}^{-k/2(1,1)} \rangle \\ &= \langle e^{\hat{b}(\phi^++\phi^-)}e^{\hat{b}(\phi^++\phi^-)}\psi^-\bar{\psi}^-e^{\hat{b}\phi^+} \rangle. \end{aligned} \quad (\text{B.8})$$

Integrating the sum of (B.7) and (B.8), we obtain

$$\langle e^{\hat{b}(\phi^++\phi^-)}e^{\hat{b}(\phi^++\phi^-)} \rangle = \frac{\sqrt{2}}{2}i\hat{b}^{-9}(2\hat{b}^2-1)\frac{\gamma(1-\hat{b}^2)^2\gamma(2-1/\hat{b}^2)}{\gamma(1-2\hat{b}^2)}\lambda^{\frac{2}{\hat{b}^2}-4}. \quad (\text{B.9})$$

Substituting in (B.5), we obtain (3.36).

References

- [1] A.M. Polyakov, *Quantum geometry of bosonic strings*, *Phys. Lett.* **B 103** (1981) 207.
- [2] E. Witten, *On string theory and black holes*, *Phys. Rev.* **D 44** (1991) 314.
- [3] V. Fateev, A. Zamolodchikov and Al. Zamolodchikov, unpublished.
- [4] V. Kazakov, I.K. Kostov and D. Kutasov, *A matrix model for the two-dimensional black hole*, *Nucl. Phys.* **B 622** (2002) 141 [[hep-th/0101011](#)].

- [5] G.W. Moore, *Gravitational phase transitions and the Sine-Gordon model*, hep-th/9203061.
- [6] S.Y. Alexandrov, V.A. Kazakov and D. Kutasov, *Non-perturbative effects in matrix models and D-branes*, *JHEP* **09** (2003) 057 [hep-th/0306177].
- [7] V. Fateev, A.B. Zamolodchikov and A.B. Zamolodchikov, *Boundary Liouville field theory. I: boundary state and boundary two-point function*, hep-th/0001012.
- [8] A.B. Zamolodchikov and A.B. Zamolodchikov, *Liouville field theory on a pseudosphere*, hep-th/0101152.
- [9] J. Park and T. Suyama, *Type 0a matrix model of black hole, integrability and holography*, *Phys. Rev. D* **71** (2005) 086002 [hep-th/0411006].
- [10] M.R. Douglas et al., *A new hat for the $c = 1$ matrix model*, hep-th/0307195:
T. Takayanagi and N. Toumbas, *A matrix model dual of type 0B string theory in two dimensions*, *JHEP* **07** (2003) 064.
- [11] K. Hori and A. Kapustin, *Duality of the fermionic 2d black hole and $N = 2$ Liouville theory as mirror symmetry*, *JHEP* **08** (2001) 045 [hep-th/0104202].
- [12] R.C. Rashkov and M. Stanishkov, *Three-point correlation functions in $n = 1$ super Liouville theory*, *Phys. Lett. B* **380** (1996) 49 [hep-th/9602148].
- [13] R.H. Poghosian, *Structure constants in the $N = 1$ super-Liouville field theory*, *Nucl. Phys. B* **496** (1997) 451 [hep-th/9607120].
- [14] T. Fukuda and K. Hosomichi, *Super Liouville theory with boundary*, *Nucl. Phys. B* **635** (2002) 215 [hep-th/0202032].
- [15] C. Ahn, C. Rim and M. Stanishkov, *Exact one-point function of $N = 1$ super-Liouville theory with boundary*, *Nucl. Phys. B* **636** (2002) 497 [hep-th/0202043].
- [16] C. Ahn, C. Kim, C. Rim and M. Stanishkov, *Duality in $N = 2$ super-Liouville theory*, *Phys. Rev. D* **69** (2004) 106011 [hep-th/0210208].
- [17] T. Eguchi and Y. Sugawara, *Modular bootstrap for boundary $N = 2$ Liouville theory*, *JHEP* **01** (2004) 025 [hep-th/0311141].
- [18] S. Ribault and V. Schomerus, *Branes in the 2-d black hole*, *JHEP* **02** (2004) 019 [hep-th/0310024].
- [19] C. Ahn, M. Stanishkov and M. Yamamoto, *Zz-branes of $N = 2$ super-Liouville theory*, *JHEP* **07** (2004) 057 [hep-th/0405274].
- [20] J. McGreevy, S. Murthy and H.L. Verlinde, *Two-dimensional superstrings and the supersymmetric matrix model*, *JHEP* **04** (2004) 015 [hep-th/0308105].
- [21] D. Israel, A. Pakman and J. Troost, *D-branes in $N = 2$ Liouville theory and its mirror*, *Nucl. Phys. B* **710** (2005) 529 [hep-th/0405259].
- [22] A. Fotopoulos, V. Niarchos and N. Prezas, *D-branes and extended characters in $SL(2, \mathbb{R})/u(1)$* , *Nucl. Phys. B* **710** (2005) 309 [hep-th/0406017].
- [23] K. Hosomichi, *$N = 2$ Liouville theory with boundary*, hep-th/0408172.
- [24] J.L. Cardy, *Conformal invariance and surface critical behavior*, *Nucl. Phys. B* **240** (1984) 514.

- [25] B. Eynard and J. Zinn-Justin, *Large order behavior of 2-d gravity coupled to $d < 1$ matter*, *Phys. Lett. B* **302** (1993) 396 [[hep-th/9301004](#)].
- [26] L.A. Pando Zayas and D. Vaman, *Condensing momentum modes in 2-d 0A string theory with flux*, [[hep-th/0507061](#)].