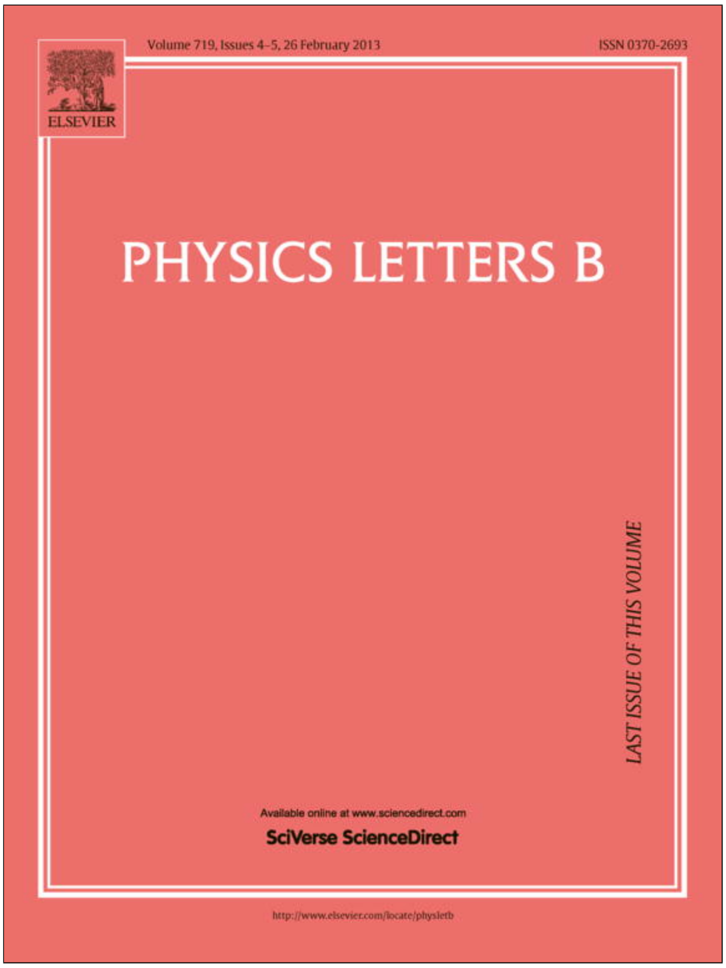
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Worldsheet S-matrix of β-deformed SYM

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ABSTRACT

We compute perturbative worldsheet S-matrix elements in the bosonic sector for β-deformed AdS/CFT in strong and weak 't Hooft coupling limits and compare with the exact S-matrix. For our purpose we take near BMN limit of TsT-transformed $AdS_5 \times S^5$ with the twisted boundary condition and compute the S-matrix on worldsheet using light-cone gauge fixed Lagrangian. For the weak coupling side, we compute the S-matrix in $SU(3)$ sector by applying coordinate Bethe ansatz method to one-loop dilatation operator obtained from the deformed super Yang–Mills theory. These analysis support the conjectured exact S-matrix in the leading order for both sides of β-deformed AdS/CFT along with the appropriate twisted boundary conditions.

1. Introduction

The S-matrix plays a key role for studying two-dimensional integrable models. With enough symmetries, the S-matrix can be determined mathematically and can be used to find particle spectrum along with exact dispersion relations and to compute finite-size effects. Based on this philosophy, there have been remarkable developments in applying the integrability methods to AdS/CFT duality between $\mathcal{N}=4$ super Yang–Mills theory (SYM) and type-IIB superstring theory on $AdS_5 \times S^5$ [1]. Exact S-matrix has been proposed [2–4] with the dressing phase [5,6], and applied to tools such as Lüscher correction [7] and thermodynamic Bethe ansatz [8].

After these successes, it is natural to extend the utility of the integrable methods to other proposed or conjectured AdS/CFT dualities. These include β-deformed SYM theory [9] which is dual to superstring theory on Lunin–Maldacena background [10] and three-parameter-deformed theory which breaks all the supersymmetry [11]. There are some clues that the deformations still maintain the integrability. First, string sigma models on the deformed backgrounds are classically integrable [12,13]. One-loop dilatation operator on the gauge theory can be mapped to integrable spin-chain models with certain twists [14]. All-loop asymptotic Bethe ansatz equations for the deformed theories were conjectured by Beisert and Roiban [15].

Another strong evidence for the integrability has come from the anomalous dimension of Konishi operator computed by twisted Lüscher formula [16] which matches with four-loop perturbative computation [17]. Related computations have been also worked out by Y-system of the β-deformed SYM [18].

With the assumption of integrability, the S-matrix and associated twisted boundary conditions have been proposed and used to derive the conjectured all-loop asymptotic Bethe ansatz equations [19]. The twisted S-matrix is given by

$$\tilde{\mathcal{S}}(p_1,p_2) = F \cdot \mathcal{S}(p_1,p_2) \cdot F,$$
$$F = e^{i\gamma_1 \Gamma}, \quad \Gamma = h \otimes \mathbb{I} \otimes \mathbb{I} \otimes h - \mathbb{I} \otimes h \otimes h \otimes \mathbb{I},$$
$$h = \mathrm{diag}\left(\frac{1}{2}, -\frac{1}{2}, 0, 0\right), \tag{1.1}$$

where the first two vector spaces in the tensor products act on fundamental representations of $SU(2|2)^2$ for particle 1 and the third and fourth on those for particle 2. The corresponding twisted boundary conditions are

$$M = e^{i(\gamma_3-\gamma_2)Jh} \otimes e^{i(\gamma_3+\gamma_2)Jh}, \tag{1.2}$$

which acts on the one-particle state.

Another support of the S-matrix conjecture comes from the strong coupling limit of the twisted AdS/CFT duality. Finite J correction of a classical giant magnon dispersion relation has been computed from the S-matrix element and twisted boundary conditions through Lüscher formula [20] and shown to match with classical sigma model computation for the γ-deformed background [21,22]. While these evidences justify the assumption of integrability, it is desirable to check the S-matrix directly either with the string theory on a deformed background in the strong coupling limit or with the $\mathcal{N}=1$ supersymmetric or non-supersymmetric gauge theories in the weak coupling limit.

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On the other hand, energy spectrum of an integrable system is determined by Bethe–Yang equations which consist of the S-matrix and the boundary conditions. Therefore, the same energy spectrum can be obtained by attributing a part of S-matrix into the boundary conditions and vice versa.

In the context of the twisted AdS/CFT, it is possible to shift the Drinfeld–Reshetikhin twist F into the boundary condition M as shown in [19]. The resulting theory is described by untwisted S-matrix while the twisted boundary condition is given by

$$M_{A,Q_1,Q_2,\ldots,Q_N} = M_A \prod_{j=1}^{N} F^2_{AQ_j}. \tag{1.3}$$

This is called "operatorial" boundary condition since it depends not only on the particle state which passes through the boundary but also on all the other states Q's in the "quantum space" which are away from the boundary. This feature is inevitable when one deals with off-diagonal S-matrix. It is shown that this combination of S-matrix and boundary conditions can produce the same "Beisert–Roiban" asymptotic Bethe ansatz equations [23].

However, these operatorial boundary conditions are difficult to realize in the perturbative computations. On string side, Frolov first showed that superstring theory on the *TsT*-transformed $AdS_5 \times S^5$ with the periodic boundary conditions is equivalent to the undeformed $AdS_5 \times S^5$ with the following twisted boundary conditions [11]:

$$\phi_i(2\pi) - \phi_i(0) = 2\pi(n_i + \epsilon_{ijk}\gamma_j J_k) \quad (i,j,k = 1,2,3). \tag{1.4}$$

Here, ϕ_i are three isometry angles of S^5 and $\gamma_j = \beta$ is a parameter for deformation of a scalar field potential in the gauge theory side and three angular momenta are given by $J_i = \int d\sigma\, \dot{\phi}_i$. Eq. (1.4) is not easy to solve for the multiparticle solutions. For the spin-chain side in the weak coupling limit, S-matrix can be computed by coordinate Bethe ansatz method. The spin-chain Hamiltonian as a dilatation operator naturally depends on the deformation parameter β in $\mathcal{N}=1$ SYM. By some nontrivial unitary transformation it can be changed into that of an untwisted spin chain as explained in [14]. However, it generates nontrivial boundary conditions which will be in general nonlocal i.e. which depends on the states on the quantum space.

For these reasons, we study the S-matrix elements in the bosonic sector of the β-deformed SYM at strong and weak 't Hooft coupling regimes which corresponds to (1.1) where the boundary condition (1.2) becomes simply a c-number. For this purpose, we consider string world-sheet action in near BMN limit and with light-cone gauge fixing which is different from Lunin–Maldacena and compute the worldsheet scattering as was done for untwisted case in [24]. In the weak coupling regime we consider one-loop dilatation operator for three-spin sector. We apply coordinate Bethe ansatz to compute one-loop S-matrix in this sector using the deformed $SU(3)$ spin-chain Hamiltonian derived in [14,25] and show that it matches with the exact Drinfeld–Reshetikhin S-matrix (1.1) in this limit.

2. Strong coupling regime: String worldsheet

The dual gravity solution of $\mathcal{N}=1$ β-deformed SYM was first constructed by Lunin and Maldacena [10]. This background could be obtained by using sequence of three *TsT*-transformations for the S^5 angles: $(\phi_1,\phi_2)_{TsT}$, $(\phi_2,\phi_3)_{TsT}$ and $(\phi_3,\phi_1)_{TsT}$ for three isometry angles with single parameter $\hat{\gamma} = \gamma\sqrt{\lambda}$, $(\gamma_i = \gamma)$. Here, $(\phi_1,\phi_2)_{TsT}$ means to take T-dualization along ϕ_1, shift $\phi_2 \to \phi_2 + \hat{\gamma}\phi_1$ and take T-dualization again for ϕ_1. As a result of *TsT*-transformation, all kinds of background fields – metric, B-fields, RR-fields and so on – are deformed or generated. If we use different parameters $\hat{\gamma}_{1,2,3}$ for each *TsT*-transformation, LM background could be generalized to three-parameter deformed background which is dual to non-supersymmetric, marginal deformed SYM. The three-parameter deformed $AdS_5 \times S^5$ spacetime metric and antisymmetric B-fields are given by the followings:

$$\begin{aligned} ds^2_{\text{string}}/R^2 &= ds^2_{AdS_5} + \sum_{i=1}^{3}\left(d\rho_i^2 + G\rho_i^2 d\tilde{\tilde{\phi}}_i^2\right) \\ &\quad + G\rho_1^2\rho_2^2\rho_3^2\left(\sum_{i=1}^{3}\hat{\gamma}_i d\tilde{\tilde{\phi}}_i\right)^2, \\ B_2 &= R^2 G\big(\hat{\gamma}_3\rho_1^2\rho_2^2 d\tilde{\tilde{\phi}}_1 \wedge d\tilde{\tilde{\phi}}_2 + \hat{\gamma}_1\rho_2^2\rho_3^2 d\tilde{\tilde{\phi}}_2 \wedge d\tilde{\tilde{\phi}}_3 \\ &\quad + \hat{\gamma}_2\rho_3^2\rho_1^2 d\tilde{\tilde{\phi}}_3 \wedge d\tilde{\tilde{\phi}}_1\big), \end{aligned} \tag{2.1}$$

where

$$G^{-1} = 1 + \hat{\gamma}_3^2\rho_1^2\rho_2^2 + \hat{\gamma}_1^2\rho_2^2\rho_3^2 + \hat{\gamma}_2^2\rho_3^2\rho_1^2. \tag{2.2}$$

There is an additional constraint $\sum_{i=1}^{3}\rho_i^2 = 1$ and three isometry angles ϕ_i have periodicity under $\sigma \to \sigma + 2\pi$. We will only consider one-parameter deformed theory $(\hat{\gamma}_i = \hat{\gamma})$ for simplicity but all discussions about string regime in this Letter are applicable even for three-parameter deformed theory.

2.1. *TsT-transformed* $AdS_5 \times S^5$ *with twisted boundary conditions*

We start from $AdS_5 \times S^5$ string with twisted boundary conditions (1.4). The nonlinear sigma model action on usual S^5 is given by

$$S = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau\, d\sigma\left[h^{\alpha\beta}\left(\partial_\alpha\rho_i\partial_\beta\rho_i + \rho_i^2\partial_\alpha\phi_i\partial_\beta\phi_i + \Lambda\left(\rho_i^2 - 1\right)\right]\right., \tag{2.3}$$

with

$$h^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}$$

Taking a *TsT*-transformation $(\phi_2,\phi_3)_{TsT}$, we obtain a new background

$$\begin{aligned} ds^2_{\text{string}}/R^2 &= ds^2_{AdS_5} + d\rho_1^2 + \rho_1^2 d\hat{\phi}_1^2 + \sum_{i=2}^{3}\left(d\rho_i^2 + \hat{G}\rho_i^2 d\hat{\phi}_i^2\right), \\ B_2 &= -\hat{\gamma}R^2\hat{G}\left(\rho_2^2\rho_3^2 d\hat{\phi}_2 \wedge d\hat{\phi}_3\right), \\ \hat{G}^{-1} &= 1 + \hat{\gamma}^2\rho_2^2\rho_3^2, \end{aligned} \tag{2.5}$$

with the background metric $\hat{G}_{ij}$ and fields $\hat{B}_{ij}$ whose non-zero components are

$$\begin{aligned} &\hat{G}_{11} = \rho_1^2, \qquad \hat{G}_{22} = \hat{G}\rho_2^2, \\ &\hat{G}_{33} = \hat{G}\rho_3^2, \qquad \hat{B}_{23} = \hat{G}\hat{\gamma}^2\rho_2^2\rho_3^2. \end{aligned} \tag{2.6}$$

Here, AdS_5 metric is defined as

$$ds^2_{AdS_5} = -\left(\frac{1+\frac{Z^2}{4}}{1-\frac{Z^2}{4}}\right)^2 dt^2 + \sum_{k=1}^{4}\frac{dZ_k\, dZ_k}{\left(1-\frac{Z^2}{4}\right)^2} \tag{2.7}$$

where $Z^2 = Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2$. Corresponding bosonic string action on the deformed S^5 is

$$\hat{S} = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau\, d\sigma \big[h^{\alpha\beta}\big(\partial_\alpha\rho_i\partial_\beta\rho_i + \rho_1^2\partial_\alpha\hat{\phi}_1\partial_\beta\hat{\phi}_1 + \hat{G}\rho_2^2\partial_\alpha\hat{\phi}_2\partial_\beta\hat{\phi}_2 + \hat{G}\rho_3^2\partial_\alpha\hat{\phi}_3\partial_\beta\hat{\phi}_3\big) - \epsilon^{\alpha\beta}\hat{B}_{\hat{\phi}_2\hat{\phi}_3}\partial_\alpha\hat{\phi}_2\partial_\beta\hat{\phi}_3 + \Lambda\big(\rho_i^2 - 1\big)\big]. \tag{2.8}$$

We will use this action to compute worldsheet S-matrix.[1]

The above TsT-transformation changes the twisted boundary conditions (1.4) to

$$\hat{\phi}_1(2\pi) - \hat{\phi}_1(0) = P_{ws},$$
$$\hat{\phi}_2(2\pi) - \hat{\phi}_2(0) = 2\pi(n_2 + \beta J_1),$$
$$\hat{\phi}_3(2\pi) - \hat{\phi}_3(0) = 2\pi(n_3 - \beta J_1), \tag{2.9}$$

where level matching condition is given by $P_{ws} = 2\pi[n_1 + \beta(J_3 - J_2)]$. This corresponds to "c-number" boundary conditions for $\hat{\phi}_2$ and $\hat{\phi}_3$ because they do not depend on J_2 and J_3.

2.2. Gauge fixed Lagrangian

To compute the string worldsheet S-matrix it is convenient to introduce new variables defined by

$$\rho_1 = \frac{1 - \frac{Y^2}{4}}{1 + \frac{Y^2}{4}}, \qquad \rho_2 = \frac{\sqrt{Y_1{}^2 + Y_2{}^2}}{1 + \frac{Y^2}{4}}, \qquad \rho_3 = \frac{\sqrt{Y_3{}^2 + Y_4{}^2}}{1 + \frac{Y^2}{4}},$$
$$\hat{\phi}_1 = \phi, \qquad \hat{\phi}_2 = \arctan(Y_2/Y_1), \qquad \hat{\phi}_3 = \arctan(Y_4/Y_3). \tag{2.10}$$

We also have to remove the redundancy from general coordinate invariance. A standard way is to consider the BMN limit [26] and its curvature corrections:

$$t \to X^+ - \frac{X^-}{2R^2}, \qquad \phi \to X^+ + \frac{X^-}{2R^2},$$
$$Z_k \to \frac{Z_k}{R}, \qquad Y_{k'} \to \frac{Y_{k'}}{R}. \tag{2.11}$$

This BMN limit simplifies the metric and B-fields as follows:

$$ds^2 = 2\,dX^+dX^- + dY^2 + dZ^2 - dX^{+2}\big(Z^2 + Y^2\big) + \frac{1}{2R^2}\big(2\,dX^- dX^+\big(Z^2 - Y^2\big) + dZ^2 Z^2 - dY^2 Y^2 + dX^+ dX^+\big(Y^4 - Z^4\big)\big),$$
$$B = \frac{1}{2R^2}\hat{\gamma}(Y_1Y_3\,dY_2 \wedge dY_4 + Y_2Y_4\,dY_1 \wedge dY_3 - Y_2Y_3\,dY_1 \wedge dY_4 - Y_1Y_4\,dY_2 \wedge dY_3). \tag{2.12}$$

Although the metric is independent of $\hat{\gamma}$ up to $1/R^2$, worldsheet scattering becomes nontrivial because the B-fields have $\hat{\gamma}$ dependence.[2]

The bosonic string Lagrangian now becomes

$$L = \frac{1}{2}h^{\alpha\beta}\big[G_{++}\partial_\alpha X^+\partial_\beta X^+ + G_{--}\partial_\alpha X^-\partial_\beta X^- + G_{+-}\big(\partial_\alpha X^+\partial_\beta X^- + \partial_\alpha X^-\partial_\beta X^+\big) + G_{Z^iZ^j}\partial_\alpha Z^i\partial_\beta Z^j + G_{Y^iY^j}\partial_\alpha Y^i\partial_\beta Y^j\big] + \frac{1}{2}\varepsilon^{\alpha\beta}\big(B_{ij}\partial_\alpha Y^i\partial_\beta Y^j + B_{+i}\partial_\alpha X^+\partial_\beta Y^i + B_{-i}\partial_\alpha X^-\partial_\beta Y^i\big), \tag{2.13}$$

with

$$\varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.14}$$

Here, α, β stand for worldsheet coordinates σ, τ and $i, j = 1, 2, 3, 4$.

As in the usual case, the Hamiltonian is just sum of Lagrange multiplier times constraint. As the epsilon coupled to anti-symmetric B-fields is a non-dynamical field, the variation of the action over worldsheet metric operate on only G-field parts.

To fix the gauge, we can use the first-order formalism which works well for undeformed theory [27–29]. First, we define the conjugate momentum

$$\mathbb{P}_\mu = \big(\gamma^{\tau\sigma}G_{\mu\nu} + B_{\mu\nu}\big)\acute{\mathbb{X}}^\nu + \gamma^{\tau\tau}G_{\mu\nu}\dot{\mathbb{X}}^\nu, \tag{2.15}$$

where we denote $\mathbb{X}^\mu = (X^+, X^-, \vec{Z}, \vec{Y})$ and $\mathbb{P}_\mu = (P_+, P_-, \vec{P}_Z, \vec{P}_Y)$; 'dot' and 'prime' as τ and σ derivatives, respectively. The resulting Hamiltonian

$$H = \frac{1}{2\gamma^{\tau\tau}}G^{\mu\nu}\bar{\mathbb{P}}_\mu\bar{\mathbb{P}}_\nu + \frac{1}{2\gamma^{\tau\tau}}G_{\mu\nu}\acute{\mathbb{X}}^\mu\acute{\mathbb{X}}^\nu - \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}\bar{\mathbb{P}}_\mu\acute{\mathbb{X}}^\mu,$$
$$\bar{\mathbb{P}}_\mu = \mathbb{P}_\mu - B_{\mu\nu}\acute{\mathbb{X}}^\nu \tag{2.16}$$

becomes zero if we impose the Virasoro constraints. Introducing the light-cone gauge $X^+ = \tau$, $P_- = \text{const}$, we can express the Lagrangian $L = \mathbb{P}_\mu\dot{\mathbb{X}}^\mu - H$ in terms of unguaged variables

$$L_{g.f.} = P_+ + \vec{P}_Y\cdot\dot{\vec{Y}} + \vec{P}_Z\cdot\dot{\vec{Z}} = \vec{P}_Y\cdot\dot{\vec{Y}} + \vec{P}_Z\cdot\dot{\vec{Z}} - H_{L.C.}, \tag{2.17}$$

where we have imposed the Virasoro constraints. The expression for the light-cone Hamiltonian is given by

$$H_{L.C.} = \tilde{H} + \frac{\hat{\gamma}}{\sqrt{\lambda}}(-\acute{Y}_1Y_2Y_3P_{Y_4} + \acute{Y}_2Y_1Y_3P_{Y_4} + \acute{Y}_1Y_2Y_4P_{Y_3} - \acute{Y}_2Y_1Y_4P_{Y_3} + \acute{Y}_3Y_4Y_1P_{Y_2} - \acute{Y}_3Y_4Y_2P_{Y_1} - \acute{Y}_4Y_3Y_1P_{Y_2} + \acute{Y}_4Y_3Y_2P_{Y_1}). \tag{2.18}$$

Here, $\tilde{H}$ is the light-cone Hamiltonian of the undeformed theory. Considering Legendre transformation and solving the equations of motion for P_{Y_i} and P_{Z_i}, we finally obtain the gauge fixed bosonic Lagrangian

$$L_{g.f.} = \frac{1}{2}\partial_\mu\vec{Z}^\dagger\cdot\partial^\mu\vec{Z} - \frac{1}{2}\vec{Z}^\dagger\cdot\vec{Z} + \frac{1}{2}\partial_\mu\vec{Y}^\dagger\cdot\partial^\mu\vec{Y} - \frac{1}{2}\vec{Y}^\dagger\cdot\vec{Y} - \mathbb{V}(\vec{Y}, \vec{Z}), \tag{2.19}$$

with $\vec{Y} = (Y_{1\dot{1}}, Y_{1\dot{2}}, Y_{2\dot{1}}, Y_{2\dot{2}})$ defined by

$$Y_{1\dot{1}} = Y_1 + iY_2, \qquad Y_{1\dot{2}} = Y_3 + iY_4,$$
$$Y_{2\dot{1}} = Y_3 - iY_4, \qquad Y_{2\dot{2}} = Y_1 - iY_2. \tag{2.20}$$

The potential term is

$$\mathbb{V} = \frac{1}{4\sqrt{\lambda}}\big[(Y_{1\dot{1}}Y_{2\dot{2}} + Y_{1\dot{2}}Y_{2\dot{1}})\big(2\partial_\sigma Y_{1\dot{1}}\partial_\sigma Y_{2\dot{2}} + 2\partial_\sigma Y_{1\dot{2}}\partial_\sigma Y_{2\dot{1}} + (\partial_\tau Z)^2 + (\partial_\sigma Z)^2\big) - Z^2\big(\partial_\tau Y_{1\dot{2}}\partial_\tau Y_{2\dot{1}} + \partial_\tau Y_{1\dot{1}}\partial_\tau Y_{2\dot{2}} + \partial_\sigma Y_{1\dot{2}}\partial_\sigma Y_{2\dot{1}} + \partial_\sigma Y_{1\dot{1}}\partial_\sigma Y_{2\dot{2}} + 2(\partial_\sigma Z)^2\big)\big] - \frac{\hat{\gamma}}{4\sqrt{\lambda}}\big[Y_{1\dot{1}}Y_{1\dot{2}}\partial_\tau Y_{2\dot{2}}\partial_\sigma Y_{2\dot{1}} - Y_{1\dot{1}}Y_{1\dot{2}}\partial_\tau Y_{2\dot{1}}\partial_\sigma Y_{2\dot{2}}$$

[1] For simplicity, we restrict ourselves to the S-matrix elements in the bosonic sector only.

[2] The same computation for original Lunin–Maldacena background shows that $(1 + \hat{\gamma}^2)$ appears in front of dY^2 which gives different masses between AdS_5 and S^5 in the gauge fixed action. This is one reason why we need to introduce twisted boundary conditions in the string theory side.

$$+ Y_{1\dot{1}} Y_{2\dot{1}} \partial_\tau Y_{1\dot{2}} \partial_\sigma Y_{2\dot{2}} - Y_{1\dot{1}} Y_{2\dot{1}} \partial_\tau Y_{2\dot{2}} \partial_\sigma Y_{1\dot{2}}$$
$$+ Y_{1\dot{2}} Y_{2\dot{2}} \partial_\tau Y_{2\dot{1}} \partial_\sigma Y_{1\dot{1}} - Y_{1\dot{2}} Y_{2\dot{2}} \partial_\tau Y_{1\dot{1}} \partial_\sigma Y_{2\dot{1}}$$
$$+ Y_{2\dot{1}} Y_{2\dot{2}} \partial_\tau Y_{1\dot{1}} \partial_\sigma Y_{1\dot{2}} - Y_{2\dot{1}} Y_{2\dot{2}} \partial_\tau Y_{1\dot{2}} \partial_\sigma Y_{1\dot{1}} \big]. \tag{2.21}$$

To compute the worldsheet S-matrix, we need to consider decompactification limit $P_- \to \infty$ in which worldsheet parameter space changes from cylinder to plane after rescaling $\sigma \to \frac{P_-}{\sqrt{\lambda}}\sigma$. Here, P_- has appeared in the integration bound for σ due to light-cone gauge fixing.

2.3. Tree-level scattering amplitudes

The string worldsheet S-matrix can be straightforwardly computed from the gauge fixed action (2.19). In the leading order of $\frac{1}{\sqrt{\lambda}}$ we define $\mathbb{T}$-matrix by

$$\mathbb{S} = \mathbb{I} + \frac{2i\pi}{\sqrt{\lambda}} \mathbb{T}. \tag{2.22}$$

We need to compute additional contribution to $\mathbb{T}$ from $\hat{\gamma}$-dependent part of $\mathbb{V}$ which contains only $\vec{Y}$. In terms of mode expansions [30]

$$Y^{1\dot{1}}(\sigma,\tau) = \int \frac{dp}{2\sqrt{\omega_p}} \big[a^{1\dot{1}}(p) e^{-i(\omega\tau - p\sigma)} + \epsilon^{12} \epsilon^{\dot{1}\dot{2}} a(p)^\dagger_{2\dot{2}} e^{i(\omega\tau - p\sigma)} \big], \tag{2.23}$$

$$Y^{1\dot{2}}(\sigma,\tau) = \int \frac{dp}{2\sqrt{\omega_p}} \big[a^{1\dot{2}}(p) e^{-i(\omega\tau - p\sigma)} + \epsilon^{12} \epsilon^{\dot{2}\dot{1}} a(p)^\dagger_{2\dot{1}} e^{i(\omega\tau - p\sigma)} \big], \tag{2.24}$$

$$Y_{1\dot{1}}(\sigma,\tau) = \int \frac{dp}{2\sqrt{\omega_p}} \big[\epsilon_{12} \epsilon_{\dot{1}\dot{2}} a^{2\dot{2}}(p) e^{-i(\omega\tau - p\sigma)} + a(p)^\dagger_{1\dot{1}} e^{i(\omega\tau - p\sigma)} \big], \tag{2.25}$$

$$Y_{1\dot{2}}(\sigma,\tau) = \int \frac{dp}{2\sqrt{\omega_p}} \big[\epsilon_{12} \epsilon_{\dot{2}\dot{1}} a^{2\dot{1}}(p) e^{-i(\omega\tau - p\sigma)} + a(p)^\dagger_{1\dot{2}} e^{i(\omega\tau - p\sigma)} \big], \tag{2.26}$$

the $\hat{\gamma}$-dependent part of $\mathbb{T}$-matrix is

$$\begin{aligned}\mathbb{T}_{\hat{\gamma}} = \hat{\gamma} \int \frac{dp\,dp'}{\Lambda(p,p')} \big[&(\omega p' - \omega' p) a(p)^\dagger_{1\dot{1}} a(p')^\dagger_{2\dot{1}} a(p)_{1\dot{1}} a(p')_{2\dot{1}} \\ &- (\omega p' - \omega' p) a(p)^\dagger_{1\dot{1}} a(p')^\dagger_{1\dot{2}} a(p)_{1\dot{1}} a(p')_{1\dot{2}} \\ &+ (\omega p' - \omega' p) a(p)^\dagger_{2\dot{2}} a(p')^\dagger_{1\dot{2}} a(p)_{2\dot{2}} a(p')_{1\dot{2}} \\ &- (\omega p' - \omega' p) a(p)^\dagger_{2\dot{2}} a(p')^\dagger_{2\dot{1}} a(p)_{2\dot{2}} a(p')_{2\dot{1}} \big].\end{aligned} \tag{2.27}$$

Here, $\omega = \sqrt{p^2+1}$ and the kinematic factor [31]

$$\Lambda(p,p') = \frac{1}{\omega' p - \omega p'}. \tag{2.28}$$

One can notice that the scattering amplitudes depend on $\hat{\gamma}$ only in YY to YY process. Explicitly, only non-zero elements of the $\mathbb{T}_{\hat{\gamma}}$ are

$$\begin{aligned}\mathbb{T}_{\hat{\gamma}} \big| Y_{1\dot{1}}(p) Y_{1\dot{2}}(p') \big\rangle &= -\hat{\gamma} \big| Y_{1\dot{1}}(p) Y_{1\dot{2}}(p') \big\rangle, \\ \mathbb{T}_{\hat{\gamma}} \big| Y_{1\dot{1}}(p) Y_{2\dot{1}}(p') \big\rangle &= +\hat{\gamma} \big| Y_{1\dot{1}}(p) Y_{2\dot{1}}(p') \big\rangle, \\ \mathbb{T}_{\hat{\gamma}} \big| Y_{2\dot{2}}(p) Y_{1\dot{2}}(p') \big\rangle &= +\hat{\gamma} \big| Y_{2\dot{2}}(p) Y_{1\dot{2}}(p') \big\rangle, \\ \mathbb{T}_{\hat{\gamma}} \big| Y_{2\dot{2}}(p) Y_{2\dot{1}}(p') \big\rangle &= -\hat{\gamma} \big| Y_{2\dot{2}}(p) Y_{2\dot{1}}(p') \big\rangle.\end{aligned} \tag{2.29}$$

Now, we consider the strong coupling limit of the exact twisted S-matrix to compare with the above tree-level amplitudes. In this limit, we can expand the twisted matrix F for small $\beta = \hat{\gamma}/\sqrt{\lambda}$[3]

$$F = e^{\frac{2\pi i \hat{\gamma}}{\sqrt{\lambda}}\Gamma} \simeq \left(\mathbb{I} + 2\pi i \hat{\gamma} \frac{\Gamma}{\sqrt{\lambda}} \right), \tag{2.30}$$

with Γ defined in (1.1) as well as the twisted S-matrix

$$\tilde{S} = \mathbb{I} + 2\pi i \frac{(2\Gamma\hat{\gamma} + \mathbb{T})}{\sqrt{\lambda}}, \tag{2.31}$$

where $\mathbb{T}$ is the undeformed matrix elements. Because the elements of the twisted S-matrix (1.1) can be written as

$$\tilde{S}_{ij} = F_{il} \mathcal{S}_{lk} F_{kj} = F_i \delta_{il} \mathcal{S}_{lk} F_k \delta_{kj} = F_i F_j \mathcal{S}_{ij}, \tag{2.32}$$

only amplitudes which are deformed in two-boson to two-boson scatterings are

$$\begin{aligned}&\tilde{S}^{(1\dot{1})(1\dot{2})}_{(1\dot{1})(1\dot{2})} = e^{-\gamma} S^{11}_{11} S^{\dot{1}\dot{2}}_{\dot{1}\dot{2}}, \qquad \tilde{S}^{(1\dot{1})(2\dot{1})}_{(1\dot{1})(2\dot{1})} = e^{+\gamma} S^{12}_{12} S^{\dot{1}\dot{1}}_{\dot{1}\dot{1}}, \\ &\tilde{S}^{(2\dot{2})(1\dot{2})}_{(2\dot{2})(1\dot{2})} = e^{+\gamma} S^{21}_{21} S^{\dot{2}\dot{2}}_{\dot{2}\dot{2}}, \qquad \tilde{S}^{(2\dot{2})(2\dot{1})}_{(2\dot{2})(2\dot{1})} = e^{-\gamma} S^{22}_{22} S^{\dot{2}\dot{1}}_{\dot{2}\dot{1}}.\end{aligned} \tag{2.33}$$

This matches with (2.29).

On the other hand, we can get the twisted boundary conditions for $\vec{Y}$ from (2.9)

$$\begin{aligned}Y_{1\dot{1}}(+\pi P_-/\sqrt{\lambda})/Y_{1\dot{1}}(-\pi P_-/\sqrt{\lambda}) &= e^{2\pi i \beta J_1}, \\ Y_{1\dot{2}}(+\pi P_-/\sqrt{\lambda})/Y_{1\dot{2}}(-\pi P_-/\sqrt{\lambda}) &= e^{-2\pi i \beta J_1}, \\ Y_{2\dot{2}}(+\pi P_-/\sqrt{\lambda})/Y_{1\dot{1}}(-\pi P_-/\sqrt{\lambda}) &= e^{2\pi i \beta J_1}, \\ Y_{2\dot{1}}(+\pi P_-/\sqrt{\lambda})/Y_{1\dot{2}}(-\pi P_-/\sqrt{\lambda}) &= e^{-2\pi i \beta J_1},\end{aligned} \tag{2.34}$$

which also agree with (1.2) with $\gamma_3 = \gamma_2 = \beta$ and $J_1 = J$.

3. Weak coupling regime: Spin chains

The spin-chain Hamiltonian corresponding to the one-loop dilatation operator of the β-deformed SYM was first studied in [14, 32]. Later, more general integrable deformation was investigated in [25]. In this section, we compute S-matrix from the spin-chain Hamiltonian using the coordinate Bethe ansatz. For simplicity, we only consider three-state spin chain which is the simplest sector with nontrivial dependence on the deformation parameter.

The one-loop dilatation operator for the three-state operators Z, X and Y is given by [14]:

$$\begin{aligned}H = \sum_{i=1}^{L} \big[&-e^{2\pi i\beta} \big(E^i_{01} E^{i+1}_{10} + E^i_{12} E^{i+1}_{21} + E^i_{20} E^{i+1}_{02} \big) \\ &- e^{-2\pi i\beta} \big(E^i_{10} E^{i+1}_{01} + E^i_{21} E^{i+1}_{12} + E^i_{02} E^{i+1}_{20} \big) \\ &+ \big(E^i_{00} E^{i+1}_{11} + E^i_{11} E^{i+1}_{22} + E^i_{22} E^{i+1}_{00} \big) \\ &+ \big(E^i_{11} E^{i+1}_{00} + E^i_{22} E^{i+1}_{11} + E^i_{00} E^{i+1}_{22} \big) \big].\end{aligned} \tag{3.1}$$

Here, the indices 0, 1, 2 stand for Z, X, Y fields, respectively and the matrix E_{ab} is defined by $E_{ab}|c\rangle = |a\rangle\delta_{bc}$. This Hamiltonian is

[3] Originally, Lunin–Maldacena background was defined for small β.

integrable because it can be obtained from Drinfeld–Reshetikhin deformation of $SU(3)$ R-matrix. For our purpose, it is more convenient to introduce a position-dependent unitary transformation [14]:

$$|n\rangle_0 \to |n\rangle_0, \qquad |n\rangle_1 \to e^{2\pi i\beta n}|n\rangle_1, \qquad |n\rangle_2 \to e^{-2\pi i\beta n}|n\rangle_2$$
$$\text{with } |n\rangle_a \equiv |\cdots 00 \overset{n}{\overset{\downarrow}{a}} 00\cdots\rangle \quad (a=0,1,2), \tag{3.2}$$

for one-particle states; these phases are multiplied for multi-particle states. Under this transformation, the Hamiltonian becomes

$$\begin{aligned} H = \sum_{i=1}^{L} \big[&-\big(E^i_{01}E^{i+1}_{10} + e^{6\pi i\beta}E^i_{12}E^{i+1}_{21} + E^i_{20}E^{i+1}_{02}\big) \\ &- \big(E^i_{10}E^{i+1}_{01} + e^{-6\pi i\beta}E^i_{21}E^{i+1}_{12} + E^i_{02}E^{i+1}_{20}\big) \\ &+ \big(E^i_{00}E^{i+1}_{11} + E^i_{11}E^{i+1}_{22} + E^i_{22}E^{i+1}_{00}\big) \\ &+ \big(E^i_{11}E^{i+1}_{00} + E^i_{22}E^{i+1}_{11} + E^i_{00}E^{i+1}_{22}\big)\big] \end{aligned} \tag{3.3}$$

along with the twisted boundary conditions

$$\begin{aligned} &|L+1\rangle_0 = |1\rangle_0, \qquad |L+1\rangle_1 = e^{-2\pi i\beta L}|1\rangle_1, \\ &|L+1\rangle_2 = e^{2\pi i\beta L}|1\rangle_2. \end{aligned} \tag{3.4}$$

To apply the coordinate Bethe ansatz, we define an one-particle state

$$|\Psi\rangle_a = \sum_{n=1}^{L} e^{ipn}|n\rangle_a \quad (a=1,2).$$

Acting the Hamiltonian (3.3) on $|\Psi\rangle$, we get the dispersion relation

$$E(p) = 4\sin^2\frac{p}{2}. \tag{3.5}$$

Now we consider the two-particle scattering amplitudes. One can find easily that the S-matrix between two particles of the same type is same as $SU(2)$ case, namely

$$s(p_1,p_2) = \frac{u_1-u_2+i}{u_1-u_2-i}, \quad u_k = \frac{1}{2}\cot\frac{p_k}{2}. \tag{3.6}$$

For the two-particle states of different types ($a \neq b$), we define

$$\begin{aligned} |\Psi\rangle &= \sum_{1\leqslant n_1\leqslant n_2\leqslant L} \big\{\Phi_{12}(n_1,n_2)|n_1,n_2\rangle_{12} + \Phi_{21}(n_1,n_2)|n_1,n_2\rangle_{21}\big\}, \\ \Phi_{12}&(n_1,n_2) \\ &= A_{12}(p_1,p_2)e^{i(p_1n_1+p_2n_2)} + A_{12}(p_2,p_1)e^{i(p_2n_1+p_1n_2)}, \\ \Phi_{21}&(n_1,n_2) \\ &= A_{21}(p_1,p_2)e^{i(p_1n_1+p_2n_2)} + A_{21}(p_2,p_1)e^{i(p_2n_1+p_1n_2)}, \end{aligned} \tag{3.7}$$

where

$$|n_1,n_2\rangle_{ab} = |\cdots 00 \overset{n_1}{\overset{\downarrow}{a}} 00\cdots 00 \overset{n_2}{\overset{\downarrow}{b}} 00\cdots\rangle. \tag{3.8}$$

In terms of these amplitudes we can define the S-matrix by

$$\begin{pmatrix} A_{12}(p_2,p_1) \\ A_{21}(p_2,p_1) \end{pmatrix} = \begin{pmatrix} \tilde{r}(p_2,p_1) & \tilde{t}(p_2,p_1) \\ \tilde{\tilde{t}}(p_2,p_1) & \tilde{\tilde{r}}(p_2,p_1) \end{pmatrix} \cdot \begin{pmatrix} A_{12}(p_1,p_2) \\ A_{21}(p_1,p_2) \end{pmatrix}. \tag{3.9}$$

From the eigenvalue equation $H|\Psi\rangle = E|\Psi\rangle$, we obtain

$$\begin{aligned} 0 = {} & A_{12}(p_1,p_2)e^{ip_2}\big(1-e^{-ip_2}-e^{ip_1}\big) \\ & + A_{12}(p_2,p_1)e^{ip_1}\big(1-e^{-ip_1}-e^{ip_2}\big) \\ & + e^{-6\pi i\beta}A_{21}(p_1,p_2)e^{ip_2} + e^{-6\pi i\beta}A_{21}(p_2,p_1)e^{ip_1}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} 0 = {} & A_{21}(p_1,p_2)e^{ip_2}\big(1-e^{-ip_2}-e^{ip_1}\big) \\ & + A_{21}(p_2,p_1)e^{ip_1}\big(1-e^{-ip_1}-e^{ip_2}\big) \\ & + e^{6\pi i\beta}A_{12}(p_1,p_2)e^{ip_2} + e^{6\pi i\beta}A_{12}(p_2,p_1)e^{ip_1}, \\ & \text{with } E = E(p_1)+E(p_2). \end{aligned} \tag{3.11}$$

From the above equations we can determine transmission and reflection coefficients as below:

$$\begin{aligned} \tilde{r}(p_1,p_2) &= \tilde{\tilde{r}}(p_1,p_2) = \frac{i}{u_1-u_2-i}, \\ \tilde{t}(p_1,p_2) &= \frac{u_1-u_2}{u_1-u_2-i}e^{6\pi i\beta}, \\ \tilde{\tilde{t}}(p_1,p_2) &= \frac{u_1-u_2}{u_1-u_2-i}e^{-6\pi i\beta}, \end{aligned} \tag{3.12}$$

and the twisted S-matrix for deformed three-spin states is given by

$$\tilde{S}_{\text{spin}} = \begin{pmatrix} s(p_1,p_2) & 0 & 0 & 0 \\ 0 & \tilde{t}(p_1,p_2) & \tilde{r}(p_1,p_2) & 0 \\ 0 & \tilde{\tilde{r}}(p_1,p_2) & \tilde{\tilde{t}}(p_1,p_2) & 0 \\ 0 & 0 & 0 & s(p_1,p_2) \end{pmatrix}. \tag{3.13}$$

The S-matrix (3.13) agrees with the weak coupling limit of the exact S-matrix (1.1) except $\tilde{t}$, $\tilde{\tilde{t}}$. This discrepancy can be attributed to the frame factors assigned differently for spin-chain and worldsheet scatterings which happens also for undeformed case [4,24]. For the β-deformed case the S-matrices for the $SU(3)$ sector are related by

$$\tilde{S}_{\text{string}} = U(p_1)\cdot\tilde{S}_{\text{spin}}\cdot U(p_2)^{-1}, \tag{3.14}$$

where the frame factor $U(p)$ is given by

$$U(p) = \begin{pmatrix} e^{ip}e^{2\pi\beta i} & 0 & 0 & 0 \\ 0 & e^{ip}e^{-2\pi\beta i} & 0 & 0 \\ 0 & 0 & e^{ip}e^{-2\pi\beta i} & 0 \\ 0 & 0 & 0 & e^{ip}e^{2\pi\beta i} \end{pmatrix}. \tag{3.15}$$

It is straightforward to check $\tilde{S}_{\text{string}}$ agrees with $\lambda \to 0$ limit of (1.1).

4. Conclusions

In this Letter we have computed worldsheet and spin-chain scatterings of the β-deformed SYM in the leading order to check validity of the proposed exact S-matrix and boundary conditions. For the strong 't Hooft coupling regime, we used the light-cone gauge fixed Lagrangian in the TsT-transformed background. We also computed weak coupling S-matrix based on $SU(3)$ spin-chain Hamiltonian. We have shown that these perturbative results match with the exact conjectures.

Here, we have considered only boson to boson scatterings in the leading order. It will be interesting to extend the checks to fermions and the higher-loop order. It will be also interesting to investigate whether our simpler background of the β-deformed theory can be more useful in finding concrete string solutions or higher correlation functions.

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