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RELATION BETWEEN YANG–BAXTER AND PAIR PROPAGATION EQUATIONS IN 16-VERTEX MODELS

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We study the relation between two integrability conditions, namely, the Yang–Baxter and the pair propagation equations, in 2D lattice models. While the two are equivalent in the 8-vertex models, discrepancies appear in the 16-vertex models. As explicit examples, we find the exactly solvable 16-vertex models which do not satisfy the Yang–Baxter equations.

1. Introduction

In the last two decades, much progress has been made in 2D integrable systems both in lattice statistical models and in continuum field theories. Recently, this progress has been associated with beautiful mathematical structures such as the universal Grassmann manifold,[1] the Kac–Moody algebra,[2] and the quantum group.[3]

In 2D lattice models, there is one approach which is based on transfer matrices (TMs) and which has been proved most successful. As Baxter showed, one can construct infinite number of commuting conserved quantities through these TMs.[4] A sufficient condition for the commutativity is that the Boltzmann weights of the 2D lattice models satisfy the famous Yang–Baxter equations (YBEs). There can obviously exist many exactly solvable models which do not satisfy YBEs. Since these are exactly solvable, one needs another scheme to solve these models if they exist.

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There is another approach, which is based on the so-called pair propagation equations (PPEs) appearing in the analysis of the algebraic Bethe ansatz. According to this method, the Boltzmann weights satisfy nonlinear coupled equations. These equations become manageable if the Boltzmann weights are defined on some algebraic curves.

In this paper, we want to study some 2D lattice models which can be exactly solvable while they do not satisfy YBEs. We are looking for our candidates from the 16-vertex models.[5,6] What we are going to show first is a relationship between YBEs and PPEs. Though YBEs and PPEs are equivalent in the 8-vertex model, discrepancies appear in the 16-vertex models. Since YBEs restrict possible candidates so strongly, PPEs can cover more exactly solvable models which do not satisfy YBEs. We give explicit examples for which we compute exact eigenvalues of transfer matrices.

2. The Pair Propagation and Conjugate Pair Propagation Equations

We follow the notation of Baxter.[4] The Boltzmann weights of the symmetric 16-vertex models are given by

$$R(\pm,\,\pm;\,\pm,\,\pm)=a,\ R(\pm,\,\mp;\,\pm,\,\mp)=b,\ R(\pm,\,\mp;\,\mp,\,\pm)=c,\ R(\pm,\,\pm;\,\mp,\,\mp)=d\,,$$
$$R(\pm,\,\mp;\,\mp,\,\mp)=e,\ R(\pm,\,\pm;\,\mp,\,\pm)=k,\ R(\pm,\,\pm;\,\pm,\,\mp)=h,\ R(\mp,\,\pm;\,\mp,\,\mp)=l\,. \tag{1}$$

The Yang–Baxter equations are given in the forms

$$\sum_{\eta,\zeta,\phi} R(\mu,\,\zeta;\,\eta,\,\beta)R'(\rho,\,\alpha;\,\phi,\,\zeta)R''(\eta,\,\phi;\,\nu,\,\sigma)$$
$$=\sum_{\eta,\zeta,\phi} R''(\mu,\,\rho;\,\eta,\,\phi)R'(\phi,\,\zeta;\,\sigma,\,\beta)R(\eta,\,\alpha;\,\nu,\,\zeta)\,. \tag{2}$$

According to the Bethe ansatz, eigenfunctions $y(\beta_1,\,\beta_2,\,\dots,\,\beta_N)$ of transfer matrices $T(v)$ for N horizontal sites become in the forms of the direct products of each variable such as $y(\beta_1,\,\beta_2,\,\dots,\,\beta_N)=g_1(\beta_1)\otimes g_2(\beta_2)\otimes\dots\otimes g_N(\beta_N)$. These are eigenfunctions of transfer matrix on the upper layer. We multiply these eigenfunctions to transfer matrices, and obtain

$$(T(v)y)_\alpha=\mathrm{Tr}(G_1(\alpha_1)\dots G_N(\alpha_N)),\quad\text{with}\quad (G_i(\alpha))_{\mu\nu}=\sum_\beta R(\mu,\,\alpha;\,\nu,\,\beta)g_i(\beta)\,. \tag{3}$$

Explicit forms of $G_i(\pm)$ are

$$G_i(+)=\begin{pmatrix} ag_i(+)+hg_i(-) & kg_i(+)+dg_i(-)\\ eg_i(+)+cg_i(-) & bg_i(+)+lg_i(-)\end{pmatrix},$$
$$G_i(-)=\begin{pmatrix} lg_i(+)+bg_i(-) & cg_i(+)+eg_i(-)\\ dg_i(+)+kg_i(-) & hg_i(+)+ag_i(-)\end{pmatrix}.$$

In order to be solved exactly, it is necessary that there exist α-independent pairs of matrices P_i, P_{i+1}, which transform $G_i(\alpha)$ into upper triangle forms

$$P_i^{-1}G_i(\alpha)P_{i+1} = H_i(\alpha) = \begin{pmatrix} g_i'(\alpha) & g_i'''(\alpha) \\ 0 & g_i''(\alpha) \end{pmatrix} . \tag{4}$$

For simplicity, we choose $\det P_i = 1$ for $i = 1, 2, \ldots, N$ and we parametrize them in the forms $P_i = \begin{pmatrix} p_i(+) & t_i(+) \\ p_i(-) & t_i(-) \end{pmatrix}$. Then Eq. (4) is written in the forms

$$G_i(\alpha)P_{i+1} = P_iH_i(\alpha) \quad \text{or} \quad P_i^{-1}G_i(\alpha) = H_i(\alpha)P_{i+1}^{-1} . \tag{5}$$

As $H_i(\alpha)$ are in upper triangle forms, we obtain

$$G_i(\alpha)\begin{pmatrix} p_{i+1}(+) \\ p_{i+1}(-) \end{pmatrix} = g_i'(\alpha)\begin{pmatrix} p_i(+) \\ p_i(-) \end{pmatrix} , \tag{6}$$

$$(-p_i(-),\ p_i(+))G_i(\alpha) = g_i''(\alpha)(-p_{i+1}(-),\ p_{i+1}(+)) . \tag{7}$$

We call Eq. (6) the pair propagation equations (I) and Eq. (7) the conjugate pair propagation equations (I). By using $R(\mu, \alpha; \nu, \beta)$, the pair propagation equations (I) are given by

$$\sum_{\beta, \nu} R(\mu, \alpha; \nu, \beta)g_i(\beta)p_{i+1}(\nu) = g_i'(\alpha)p_i(\mu) . \tag{8}$$

Explicit forms of these equations are

$$\begin{cases} (ag_i(+) + hg_i(-))p_{i+1}(+) + (kg_i(+) + dg_i(-))p_{i+1}(-) = g_i'(+)p_i(+) , \\ (lg_i(+) + bg_i(-))p_{i+1}(+) + (cg_i(+) + eg_i(-))p_{i+1}(-) = g_i'(-)p_i(+) , \\ (eg_i(+) + cg_i(-))p_{i+1}(+) + (bg_i(+) + lg_i(-))p_{i+1}(-) = g_i'(+)p_i(-) , \\ (dg_i(+) + kg_i(-))p_{i+1}(+) + (hg_i(+) + ag_i(-))p_{i+1}(-) = g_i'(-)p_i(-) , \end{cases} \tag{9}$$

while the conjugate pair propagation equations (I) are given by

$$\sum_{\beta, \mu} R(\mu, \alpha; \nu, \beta)g_i(\beta)q_i(\mu) = g_i''(\alpha)q_{i+1}(\nu) , \tag{10}$$

where we have used the notation $q_i(+) = -p_i(-)$, $q_i(-) = p_i(+)$. Explicit forms of these equations are obtained from Eq. (9) by replacing $g_i'(\pm) \to g_i''(\pm)$, $p_{i+1}(+) \to -p_i(-)$, $p_{i+1}(-) \to p_i(+)$, $p_i(+) \to -p_{i+1}(-)$, $p_i(-) \to p_{i+1}(+)$, $c \leftrightarrow d$, and $e \leftrightarrow k$.

Next we consider the second type of the pair propagation equations, which we call the pair propagation equations (II). If the models are exactly solvable by using eigenfunctions $y(\beta_1, \beta_2, \ldots, \beta_N) = g_1(\beta_1) \otimes g_2(\beta_2) \otimes \cdots \otimes g_N(\beta_N)$ acting on the upper layer of the transfer matrices, they are exactly solvable by using eigenfunction

$\tilde{y}(\alpha_1, \alpha_2, \ldots, \alpha_N) = \tilde{g}_1(\alpha_1) \otimes \tilde{g}_2(\alpha_2) \otimes \cdots \otimes \tilde{g}_N(\alpha_N)$ acting on the lower layer. Similar equations corresponding to Eqs. (3) and (4) are given by

$$(\tilde{y}T(v))_\beta = \mathrm{Tr}(\tilde{G}_1(\beta_1) \cdots \tilde{G}_N(\beta_N)), \quad \text{with} \quad (\tilde{G}_i(\beta))_{\mu\nu} = \sum_\alpha R(\mu, \alpha; \nu, \beta)\tilde{g}_i(\alpha), \tag{11}$$

and

$$\tilde{P}_i^{-1}\tilde{G}_i(\beta)\tilde{P}_{i+1} = \tilde{H}_i(\alpha) = \begin{pmatrix} \tilde{g}_i'(\beta) & \tilde{g}_i'''(\beta) \\ 0 & \tilde{g}_i''(\beta) \end{pmatrix}. \tag{12}$$

The pair propagation equations (II) are given by

$$\sum_{\alpha,\nu} R(\mu, \alpha; \nu, \beta)\tilde{g}_i(\alpha)\tilde{p}_{i+1}(\nu) = \tilde{g}_i'(\beta)\tilde{p}_i(\mu). \tag{13}$$

By the symmetry of the Boltzmann weights, explicit forms of these pair propagation equations (II) are obtained from the pair propagation equations (I) by replacing untilde variables by tilde variables and $c \leftrightarrow d$, $h \leftrightarrow l$. Similarly we obtain the conjugate pair propagation equations (II) from Eq. (10) by the same replacement.

Our strategy to solve the pair propagation equations is the following. These pair propagation and conjugate pair propagation equations are special bilinear equations of four variables such as $g_i(\pm)$, $g_i'(\pm)$, $p_i(\pm)$, and $p_{i+1}(\pm)$, and it is difficult to solve these equations directly. Then we first derive nonlinear equations, where only two ratios of the variables such as $r_i = p_i(-)/p_i(+)$ and $r_{i+1} = p_{i+1}(-)/p_{i+1}(+)$ appear. Instead of solving four coupled equations in the pair propagation equations, we first solve these four nonlinear equations. For each solution of these nonlinear but two variable equations, only one of the four equations of the pair propagation equations is independent, and from that we can obtain the eigenvalues of the transfer matrices.

From the condition that there exists nontrivial solutions for $g_i(\pm)$ and $g_i'(\pm)$ in Eq. (9), we obtain

$$r_i^2 + r_{i+1}^2 - \Gamma_1(r_i^2 r_{i+1}^2 + 1) - \Gamma_2 r_i r_{i+1} + \Gamma_3 r_i(1 + r_{i+1}^2) + \Gamma_4 r_{i+1}(1 + r_i^2) = 0, \tag{14}$$

where

$$\begin{cases} \Gamma_1 = (cd - ek)/(ab - hl), \\ \Gamma_2 = (a^2 + b^2 + e^2 + k^2 - c^2 - d^2 - h^2 - l^2)/(ab - hl), \\ \Gamma_3 = (cl + dh - ak - be)/(ab - hl), \\ \Gamma_4 = (ae + bk - ch - dl)/(ab - hl), \end{cases}$$

and

$$r_i = p_i(-)/p_i(+), \quad r_{i+1} = p_{i+1}(-)/p_{i+1}(+).$$

From the condition that there exists nontrivial solutions for $p_i(\pm)$ and $p_{i+1}(\pm)$, we obtain

$$s_i^2 + s_i'^2 - \Gamma_5(s_i^2 s_i'^2 + 1) - \Gamma_6 s_i s_i' + \Gamma_7 s_i'(1 + s_i^2) + \Gamma_8 s_i(1 + s_i'^2) = 0, \tag{15}$$

where

$$\begin{cases} \Gamma_5 = (cd - hl)/(ab - ek), \\ \Gamma_6 = (a^2 + b^2 + h^2 + l^2 - c^2 - d^2 - e^2 - k^2)/(ab - ek), \\ \Gamma_7 = (ce + dk - ah - bl)/(ab - ek), \\ \Gamma_8 = (al + bh - ck - de)/(ab - ek), \end{cases}$$

and

$$s_i = g_i(-)/g_i(+), \quad s'_i = g'_i(-)/g'_i(+).$$

Similarly, from the condition that there exists nontrivial solutions for $g'_i(\pm)$ and $p_{i+1}(\pm)$, we obtain

$$r_i^2 + s_i^2 - \Gamma_9(r_i^2 s_i^2 + 1) - \Gamma_{10} r_i s_i + \Gamma_{11} r_i (1 + s_i^2) + \Gamma_{12} s_i (1 + r_i^2) = 0, \qquad (16)$$

where

$$\begin{cases} \Gamma_9 = (bd - eh)/(ac - kl), \\ \Gamma_{10} = (a^2 + c^2 + e^2 + h^2 - b^2 - d^2 - k^2 - l^2)/(ac - kl), \\ \Gamma_{11} = (bl + dk - ah - ce)/(ac - kl), \\ \Gamma_{12} = (ae + ch - bk - dl)/(ac - kl). \end{cases}$$

Finally, from the condition that there exists nontrivial solutions for $g_i(\pm)$ and $p_i(\pm)$, we obtain

$$s_i'^2 + r_{i+1}^2 - \Gamma_{13}(s_i'^2 r_i^2 + 1) - \Gamma_{14} s'_i r_{i+1} + \Gamma_{15} s'_i (1 + r_{i+1}^2) + \Gamma_{16} r_{i+1} (1 + s_i'^2) = 0, \qquad (17)$$

where

$$\begin{cases} \Gamma_{13} = (bd - kl)/(ac - eh), \\ \Gamma_{14} = (a^2 + c^2 + k^2 + l^2 - b^2 - d^2 - e^2 - h^2)/(ac - eh), \\ \Gamma_{15} = (be + dh - ak - cl)/(ac - eh), \\ \Gamma_{16} = (al + ck - bh - de)/(ac - eh). \end{cases}$$

We obtain the conjugate pair propagation equations (II) from the pair propagation equations (I) by replacing $p_i,\ p_{i+1},\ g_i,\ g'_i \to \tilde{p}_i,\ \tilde{p}_{i+1},\ \tilde{g}_i,\ \tilde{g}'_i$, that is, by replacing ratios $r_i,\ r_{i+1},\ s_i,\ s'_i \to \tilde{r}_i,\ \tilde{r}_{i+1},\ \tilde{s}_i,\ \tilde{s}'_i$ and $c \leftrightarrow d,\ h \leftrightarrow l$. Explicit forms are given by

$$\tilde{r}_i^2 + \tilde{r}_{i+1}^2 - \Gamma_1(\tilde{r}_i^2 \tilde{r}_{i+1}^2 + 1) - \Gamma_2 \tilde{r}_i \tilde{r}_{i+1} + \Gamma_3 \tilde{r}_i (1 + \tilde{r}_{i+1}^2) + \Gamma_4 \tilde{r}_{i+1} (1 + \tilde{r}_i^2) = 0, \qquad (18)$$

$$\tilde{s}_i^2 + \tilde{s}_i'^2 - \Gamma_5(\tilde{s}_i^2 \tilde{s}_i'^2 + 1) - \Gamma_6 \tilde{s}_i \tilde{s}'_i - \Gamma_8 \tilde{s}'_i (1 + \tilde{s}_i^2) - \Gamma_7 \tilde{s}_i (1 + \tilde{s}_i'^2) = 0. \qquad (19)$$

3. Connection between the Yang–Baxter and the Pair Propagation Equations

Next we consider the connection between the Yang–Baxter and the pair propagation equations. We consider products of three R matrices in Eq. (2) as matrices with row indexed by β, μ, ρ and with column indexed by α, ν, σ. We denote quantities in the left-hand side as $A(\beta,\ \mu,\ \rho|\alpha,\ \nu,\ \sigma)$ and those in the right-hand side as

$B(\beta,\,\mu,\,\rho|\alpha,\,\nu,\,\sigma)$. To show the relation $A(\beta,\,\mu,\,\rho|\alpha,\nu,\,\sigma) = B(\beta,\,\mu,\,\rho|\alpha,\,\nu,\,\sigma)$ is equivalent to show

$$\sum_{\alpha,\nu,\rho} A(\beta,\,\mu,\,\rho|\alpha,\,\nu,\,\sigma)v_1(\alpha)v_2(\nu)v_3(\sigma) = \sum_{\alpha,\nu,\rho} B(\beta,\,\mu,\,\rho|\alpha,\,\nu,\sigma)v_1(\alpha)v_2(\nu)v_3(\sigma) \tag{20}$$

for three arbitrary vectors $v_1(\alpha)v_2(\nu)v_3(\sigma)$. Explicit forms of these equations are given by

$$\begin{aligned}\sum_{\eta,\zeta,\phi,\alpha,\nu,\sigma} & R(\mu,\,\zeta;\,\eta,\,\beta)R'(\rho,\,\alpha;\,\phi,\,\zeta)R''(\eta,\,\phi;\,\nu,\,\sigma)v_1(\alpha)v_2(\nu)v_3(\sigma)\\ &= \sum_{\eta,\zeta,\phi,\alpha,\nu,\sigma} R''(\mu,\,\rho;\eta,\,\phi)R'(\phi,\,\zeta;\,\sigma,\,\beta)R(\eta,\,\alpha;\,\nu,\zeta)v_1(\alpha)v_2(\nu)v_3(\sigma)\,.\end{aligned} \tag{21}$$

We can transform these by using the pair propagation equations as

$$\begin{aligned}(\text{left-hand side}) &\equiv \sum_{\eta,\zeta,\phi,\alpha} R(\mu,\,\zeta;\,\eta,\,\beta)R'(\rho,\,\alpha;\,\phi,\,\zeta)v_1(\alpha)u_2'(\eta)u_3'(\phi)\\ &\equiv \sum_{\eta,\zeta} R(\mu,\,\zeta;\,\eta,\,\beta)t_1'(\zeta)u_2'(\eta)t_3'(\rho)\\ &\equiv z_1'(\beta)z_2'(\mu)t_3'(\rho)\,,\end{aligned} \tag{22}$$

$$\begin{aligned}(\text{right-hand side}) &\equiv \sum_{\eta,\zeta,\phi,\sigma} R''(\mu,\,\rho;\,\eta,\,\phi)R'(\phi,\,\zeta;\,\sigma,\,\beta)u_1''(\zeta)u_2''(\eta)v_3(\sigma)\\ &\equiv \sum_{\eta,\phi} R''(\mu,\,\rho;\,\eta,\,\phi)t_1''(\beta)u_2''(\eta)t_3''(\phi)\\ &\equiv t_1''(\beta)z_2''(\mu)z_3''(\rho)\,.\end{aligned} \tag{23}$$

Ratios of vectors change in the following way:

$$(\text{left-hand side}):\begin{cases} v_1(-)/v_1(+) \xrightarrow[\text{(II)}]{} t_1'(-)/t_1'(+) \xrightarrow[\text{(I)}]{} z_1'(-)/z_1'(+)\,,\\ v_2(-)/v_2(+) \xrightarrow[\text{(I)}]{} u_2'(-)/u_2'(+) \xrightarrow[\text{(I)}]{} z_2'(-)/z_2'(+)\,,\\ v_3(-)/v_3(+) \xrightarrow[\text{(III)}]{} u_3'(-)/u_3'(+) \xrightarrow[\text{(III)}]{} t_3'(-)/t_3'(+)\,,\end{cases} \tag{24}$$

$$(\text{right-hand side}):\begin{cases} v_1(-)/v_1(+) \xrightarrow[\text{(I)}]{} u_1''(-)/u_1''(+) \xrightarrow[\text{(II)}]{} t_1''(-)/t_1''(+)\,,\\ v_2''(-)/v_2''(+) \xrightarrow[\text{(I)}]{} u_2''(-)/u_2''(+) \xrightarrow[\text{(I)}]{} z_2''(-)/z_2''(+)\,,\\ v_3''(-)/v_3''(+) \xrightarrow[\text{(III)}]{} t_3''(-)/t_3''(+) \xrightarrow[\text{(III)}]{} z_3''(-)/z_3''(+)\,,\end{cases} \tag{25}$$

where we have used Eqs. (I), (II), and (III), which connect *in* variable X with *out* variable Y in the following forms:

$$\begin{cases}(\text{I}): X^2+Y^2-\Gamma_1(X^2Y^2+1)-\Gamma_2XY+\Gamma_3Y(1+X^2)+\Gamma_4X(1+Y^2)=0\,,\\ (\text{II}): X^2+Y^2-\Gamma_5(X^2Y^2+1)-\Gamma_6XY+\Gamma_7Y(1+X^2)+\Gamma_8X(1+Y^2)=0\,,\\ (\text{III}): X^2+Y^2-\Gamma_5(X^2Y^2+1)-\Gamma_6XY-\Gamma_8Y(1+X^2)-\Gamma_7X(1+Y^2)=0\,.\end{cases}$$

If the forms of Eqs. (I) and (II) are the same, Eq. (21) is satisfied. Conditions that the forms of Eqs. (I) and (II) are the same lead to $\Gamma_i = \Gamma_{i+4} (i = 1 - 4)$. We call these conditions the candidate conditions to satisfy the Yang–Baxter equations, because these conditions do not guarantee to satisfy the Yang–Baxter equations but mean to satisfy three vectors multiplied forms of the Yang–Baxter equations.

In the 16-vertex model case, the above candidate conditions lead to further restrictions on the Boltzmann weights, that is, we obtain two possibilities: (i) $a+d = b+c,\ e = h,\ k = l$, and (ii) $e = l,\ k = h$. Taking into account of these considerations to find the candidates, we consider the following exactly solvable cases, which are the more restricted cases than the above possibilities: (i) $a = c,\ b = d,\ e = h,\ k = l$, and (ii) $a = d,\ b = c,\ e = l,\ k = h$ in the next section.

4. Exactly Solvable Cases in 16-Vertex Models

The 16-vertex model has not been solved exactly in the whole regime, and has not yet been shown to be integrable. In this situation, it seems to us that the 16-vertex model will not be integrable in the whole regime. Therefore, we will restrict ourselves in the following to more specialized cases, to expect to find integrable cases. Furthermore, since the main point of our paper is to establish the relation between Yang–Baxter and pair propagation equations, it is our interest to examine only special integrable cases of the original model in order to demonstrate the method of Sec. 3 instead of analyzing the whole regime completely.

(i) $a = c,\ b = d,\ e = h,\ k = l$ *case*

In this case, we find that the Yang–Baxter equations are satisfied in a sense that there always exists a nontrivial set $\{a'', , b'', , e'', k''\}$ for given sets $\{a, , b, , e, k\}$ and $\{a', , b', , e', k'\}$. By explicit calculations, apparent independent Yang–Baxter equations reduce from $32 - 4 = 28$ to 3 in the forms

$$\begin{cases} e''C_\alpha - k''C_\beta = 0\,, \\ (a'' - b'')C_\beta - e''C_\delta = 0\,, \\ a''C_\xi - b''C_\eta = 0\,, \end{cases} \tag{26}$$

where $C_\alpha = ak' + a'k + be' + b'e$, $C_\beta = ae' + a'e + bk' + b'k$, $C_\delta = (a-b)(a'-b') + (e-k)(e'-k')$, $C_\eta = aa' + bb' + ee' + kk'$, and $C_\xi = ab' + a'b + ek' + e'k$. Then for given sets $\{a, , b, , e, k\}$ and $\{a', , b', , e', k'\}$, there always exists a nontrivial set $\{a'', , b'', , e'', k''\}$, that is, the Yang–Baxter equations are always satisfied.

In this case, Eqs. (14) to (17) give

$$\begin{cases} (r_i^2 - 1)(r_{i+1}^2 - 1) = 0\,, \\ (s_i^2 - 1)(s_i'^2 - 1) = 0\,, \\ r_i^2 + s_i^2 - \Gamma_9(r_i^2 s_i^2 + 1) - \Gamma_{10} r_i s_i + \Gamma_{11}[r_i(1 + s_i^2) - s_i(1 + r_i^2)] = 0\,, \\ s_i'^2 + r_{i+1}^2 - \Gamma_{13}(s_i'^2 r_{i+1}^2 + 1) - \Gamma_{14} r_i s_i + \Gamma_{15}[s_i'(1 + r_{i+1}^2) - r_{i+1}(1 + s_i'^2)] = 0\,, \end{cases} \tag{27}$$

where

$$\begin{cases} \Gamma_9 = (b^2 - e^2)/(a^2 - k^2), \\ \Gamma_{10} = 2(a^2 + e^2 - b^2 - k^2)/(a^2 - k^2), \\ \Gamma_{11} = 2(bk - ae)/(a^2 - k^2), \\ \Gamma_{13} = (b^2 - k^2)/(a^2 - e^2), \\ \Gamma_{14} = 2(a^2 + k^2 - b^2 - e^2)/(a^2 - e^2), \\ \Gamma_{15} = 2(be - ak)/(a^2 - e^2). \end{cases}$$

Solutions are the combinations of $r_i = s_i = \pm 1$ and $r_{i+1} = s_i' = \pm 1$ (signs are independent for both). Then we obtain the following cases:

(ia) $r_i = s_i = r_{i+1} = s_i' = \pm 1$ *case*
Choosing $p_i = p_{i+1}$, the eigenfunctions at this site become

$$g_i = \begin{pmatrix} g_i(+) \\ \pm g_i(+) \end{pmatrix}, \quad g_i' = [a + b \pm (e + k)]g_i, \; g_i'' = 0. \tag{28}$$

(ib) $r_i = s_i = -r_{i+1} = -s_i' = \pm 1$ *case*
Choosing $p_i = p_{i+1}$, the eigenfunctions become

$$g_i = \begin{pmatrix} g_i(+) \\ \pm g_i(+) \end{pmatrix}, \; g_i' = [a - b \pm (e - k)] \begin{pmatrix} g_i(+) \\ \mp g_i(+) \end{pmatrix}, \; g_i'' = 0. \tag{29}$$

When we mix these cases, it is necessary to take $r_1 = r_{N+1}$ for periodicity, but otherwise we can mix (ia) and (ib) in such a way that the number of times of mixing $r_i = -r_{i+1} = 1$ cases = the number of times of mixing $r_i = -r_{i+1} = -1$ cases. General eigenvalues of transfer matrices give

$$\Lambda = (a + b + e + k)^{m_1}(a + b - e - k)^{m_2}[(a - b)^2 - (e - k)^2]^{m_3}(\pm 1)^{m_3} \tag{30}$$

by using non-negative integers m_1, m_2, and m_3, where $m_1 + m_2 + 2m_3 = N$, and $m_1 = 0$ or $m_2 = 0$ must be satisfied in the case $m_3 = 0$.

From explicit expression of transfer matrices at small N, we obtain $\Lambda = a + b \pm (e+k)$ for $N = 1$ and $\Lambda = (a+b+e+k)^2$, $(a+b-e-k)^2$, $\pm(a-b+e-k)(a-b-e+k)$ for $N = 2$, which agree with the above formula.

(ii) $a = d, \; b = c, \; e = l, \; k = h$ *case*
In this case, the Yang–Baxter equations are not satisfied but transfer matrix commute. We first explain why the Yang–Baxter equations are not satisfied in this case through the 8-vertex case, because the expression becomes rather complicated in the 16-vertex case, but the mechanism is the same. Then we consider this special 8-vertex case, that is, $a = d, \; b = c, \; e = h = k = l = 0$, and the explicit Yang–Baxter equations give

$$ab'a'' + aa'a'' = ab'a'' + ba'b'' , \tag{31}$$

$$ab'b'' + aa'b'' = ba'b'' + bb'b'' , \tag{32}$$

$$\cdots .$$

From Eqs. (31) and (32), we obtain $(a'+b')(a''-b'')=0$, but as $a'+b'\neq 0$ because Boltzmann weights must be positive, and as is in general $a''\neq b''$, the Yang–Baxter equations are not satisfied in general. The same mechanism happens in this 16-vertex case, and as is in general $a+d\neq b+c,\ e+l\neq k+h$, the Yang–Baxter equations are not satisfied in this 16-vertex case.

In this 16-vertex case, Eqs. (14) to (17) give

$$\begin{cases} (r_i^2-1)(r_{i+1}^2-1)=0\,, \\ (s_i'^2-1)(s_i^2-1)=0\,, \\ (r_i^2-1)(s_i^2-1)=0\,, \\ (r_{i+1}^2-1)(s_i'^2-1)=0\,. \end{cases} \tag{33}$$

Then we substitute the solutions of Eq. (33) into the original pair propagation and conjugate pair propagation equations, and we obtain the following cases:

(iia) $r_i=r_{i+1}=-s_i=-s_i'=\pm 1$ *case*
Choosing $p_i=p_{i+1}$, the eigenfunctions at this site become

$$g_i=\begin{pmatrix} g_i(+) \\ \mp g_i(+) \end{pmatrix},\ g_i'=0,\ g_i''=[a+b\mp(e+k)]g_i\,. \tag{34}$$

(iib) $r_i=-r_{i+1}=s_i=-s_i'=\pm 1$ *case*
In this case, choosing $p_i=p_{i+1}$, the eigenfunctions at this site become

$$g_i=\begin{pmatrix} g_i(+) \\ \pm g_i(+) \end{pmatrix},\ g_i'=0,\ g_i''=[-a+b\pm(e-k)]\begin{pmatrix} g_i(+) \\ \mp g_i(+) \end{pmatrix}. \tag{35}$$

If we mix these cases, we obtain exactly the same formula Eq. (30). From the explicit expression of the transfer matrix of small N, we obtain $\Lambda=a+b\pm(e+k)$ for $N=1$ and $\Lambda=(a+b+e+k)^2$, $(a+b-e-k)^2$, $\pm(a-b+e-k)(a-b-e+k)$ for $N=2$, which agree with this formula.

5. Summary and Discussion

We have clarified the connection between the Yang–Baxter and the pair propagation equations in the 16-vertex models. In the 16-vertex models, we find exactly solvable example of (i) $a=c,\ b=d,\ e=h,\ k=l$ case, where the Yang–Baxter equations are satisfied.

We find another exactly solvable example of (ii) $a=d,\ b=c,\ e=l,\ k=h$ case. By explicit calculation, we can find that conditions $a+d=b+c,\ e+l=k+h$ are necessary to satisfy the Yang–Baxter equations in this case, but these are not satisfied in general, that is, the Yang–Baxter equations are not the necessary condition for the solvability. Though the Yang–Baxter equations are not satisfied, we can show that transfer matrices commute (integrable) for any lattice size N, which will be discussed in a separate paper.

In the 16-vertex models, from these exactly solvable examples, integrable cases which satisfy the Yang–Baxter equations are rather limited cases in the whole exactly solvable cases. In this sense, the pair propagation equations are more fundamental, and even if the Yang–Baxter equations are not satisfied, if the pair propagation equations are solvable, it is sufficient for our purpose to find eigenvalues of transfer matrices.

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