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FRACTIONAL SUPERSYMMETRIES IN PERTURBED COSET CFTs AND INTEGRABLE SOLITON THEORY

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We study integrable perturbations of the coset CFTs. The models are characterized by two fractional supersymmetries that are dual to each other. Generally, these models can be considered as restrictions of new integrable field theories we call fractional super soliton field theories. We study the connections with other models such as perturbations of WZW models, super sine-Gordon theory, Gross–Neveu models, and principal chiral models.

1. Introduction

In the short-distance limit of a $(1 + 1)$-dimensional quantum field theory (QFT), the mass scale of the model becomes irrelevant, and the theory is governed by a conformally invariant quantum field theory (CFT) [1]. More generally, CFT describes the behavior of QFT at a renormalization-group fixed point. Thus the classification of CFT provides a classification of all possible types of short distance structure. Given a conformal field theory, there is no unique massive theory with this prescribed behavior at short distances. However, an interesting problem is formulated by requiring the massive theory to be integrable. One can contemplate classifying integrable QFT via their short-distance structure. As we will see, this is a rather fruitful point of view, as it will lead to many new classes of integrable QFT.

It has been conjectured that all rational CFTs can be realized through a coset construction of Wess–Zumino–Witten (WZW) models and orbifolds. Denote the level-K WZW model for the simple Lie algebra G as G_K. We will only consider the coset CFTs of the form $G_K \otimes G_L/G_{K+L}$. Our aim is to associate an integrable QFT to each such coset.

There are primarily two approaches to the stated problem. One method, initiated by Zamolodchikov [2] is to consider CFT perturbed by certain relevant operators. For some choices of the perturbing operator, it can be demonstrated

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that there are some additional nontrivial integer spin conserved currents, and this is taken to be a sign that the perturbed theory is integrable. The other method is to begin with a QFT that is known to be integrable, and from it to obtain new theories by a special restriction of the Hilbert space that preserves the integrability [3–7]. The restriction is a massive analog of the Feigin–Fuchs construction, and in fact reduces to it in the massless limit. In this latter method, the spectrum of particles of the new theories can be deduced from the restriction of the spectrum of the original model.

Let us briefly summarize the known results. The case studied in greatest detail is the coset $SU(2)_1 \otimes SU(2)_L / SU(2)_{1+L}$ perturbed by the operator of dimension $(L+1)/(L+3)$, for arbitrary L. This model is an integrable restriction of the sine-Gordon theory (RSG). The exact spectrum and S-matrices can be found in refs. [5, 6]. In an analogous fashion, it has been conjectured [3] that perturbations of the coset $G_1 \otimes G_L / G_{1+L}$ by the operator with dimension $(L+1)/(L+h^*+1)$ (h^* is the dual Coxeter number of G) have an S-matrix that is related to the restricted $\mathscr{R}$-matrix of the Toda field theory based on the untwisted affine Kăc–Moody algebra $G^{(1)}$, henceforth denoted Toda ($G^{(1)}$). (Toda ($SU(2)^{(1)}$) ≡ sine-Gordon theory.) Connections between the affine Toda theories and perturbations of CFT were also suggested in refs. [8, 9]; these authors' investigation was at the level of equations of motion, where the necessary restriction of the Hilbert space is not apparent.

It should be pointed out that some special cases of perturbations of the cosets $G_1 \otimes G_1 / G_2$ were studied without the formalism of restriction in refs. [10–12]. See refs. [11, 49] for other references and a review. Our understanding of these results indicates that they are best understood as a limiting case of the restriction formalism. In fact, the connection of these models to soliton equations can only be correctly formulated via restriction. Let us clarify this statement. From the viewpoint of the restriction formalism, for the special case of perturbations of $G_1 \otimes G_1 / G_2$ the degrees of freedom of the restricted model are "frozen" out and yield a nondegenerate mass spectrum of r-particles, where one particle is associated with each fundamental representation ($r = \text{rank}(G)$). In the case of $G = SU(2)$, this phenomenon was explained in refs. [5, 6] where it was shown to yield the correct spectrum of the perturbed $SU(2)_1 \otimes SU(2)_1 / SU(2)_2$ theory, i.e. a single massive Majorana fermion. In general the mass spectrum is equivalent to the spectrum of masses that follows from the lagrangian of the Toda ($G^{(1)}$) theory (diagonalization of the mass term). However these particles are not to be identified with the r Toda fields themselves, even if classically they have the same mass. For this special case, the S-matrices simply follow from crossing, unitarity, and the bootstrap. We refer the reader to the results of sect. 7 on the $SU(N)$ cosets for an explicit realization of these remarks.

A novel feature of the RSG theories is their invariance under symmetries that generalize supersymmetry to a fractional supersymmetry [6, 13]. For the RSG theories associated with the coset $SU(2)_1 \otimes SU(2)_L/SU(2)_{1+L}$, there are two conserved charges of Lorentz spin $\pm 2/(L+2)$. These symmetries were found to commute with the exact S-matrix, and were also constructed off-shell in perturbation theory. Similar results were found by Zamolodchikov for $L = 2, 4$ [13].

Our intention in the present work is to determine the general pattern for perturbations of the coset CFT $G_K \otimes G_L/G_{K+L}$. The main features that can be concluded from the results of this paper are as follows. The coset CFTs are invariant under the duality transformation $K \leftrightarrow L$. We find perturbations of the coset models that preserve the conservation of some fractional Lorentz spin currents. There are two kinds of fractional spin symmetries that are dual to each other, which we will call $Q^{(K)}$ and $Q^{(L)}$ symmetries. For fixed K the series of massive models obtained by varying L all have the same $Q^{(K)}$ symmetry; i.e. there are conserved charges in each model of the series with Lorentz spin independent of L and equal to $h^*/(K+h^*)$. Because the perturbations preserve the duality, there also exist symmetries $Q^{(L)}$ of Lorentz spin $h^*/(L+h^*)$. It was the $Q^{(L)}$ type of symmetry that was discovered in the RSG theories in ref. [6], where they were shown to be fractional supersymmetries. Consider now the case where G is simply-laced. For fixed K but variable L, the perturbed coset models can again be derived as a restriction of an integrable soliton theory. However since the r ($=$ rank(G)) bosonic fields of the $K = 1$ series must be augmented by fermions and their parafermionic generalizations in the $K > 1$ generalized Feigin–Fuchs (FF) construction, the massive soliton field theory also contains these extra fields. These fields are a complete set for the massive theory. We will generically refer to this set of fundamental fields as the FF fields. Furthermore this new augmented theory manifests the fractional $Q^{(K)}$ symmetry, and thus may be considered as a kind of integrable fractional super soliton theory. This represents a new class of integrable QFT. Consider now the case where G is nonsimply laced. Now the generalized FF construction (or vertex operator construction) for even the $G_1 \otimes G_L/G_{1+L}$ theories requires additional nonbosonic fields [14–16]; in the case of $G = B_N$ one additional fermion is needed for the short root. We find that integrable perturbations of the $(B_N)_1 \otimes (B_N)_L/(B_N)_{1+L}$ series (WB_N-series) are related to Toda theory on affine super Lie algebras. We will not present a completely general theory here, but will motivate the above general scheme with some specific examples.

Our construction has some interesting new consequences for some previously known QFTs. The $L \to \infty$ limit of the perturbed coset models yields a current–current perturbation of the WZW models, which are closely related to the principal chiral models (sigma models) with Wess–Zumino term. The perturbed coset construction in this limit provides a new solution to the soliton spectrum and S-matrices of these models. Furthermore, these perturbations of WZW models are seen to possess hidden fractional supersymmetries. Also, the perturbations of the SU(N) cosets for $K = 1$ and $L \to \infty$ give the SU(N) Gross–Neveu models. Finally the double limit $K, L \to \infty$ yields the principal chiral models (without Wess–Zumino term).

Our conjectured S matrices for the SU(N) cosets indicate that the $K=1$ series of models are NOT restrictions of the affine Toda ($G^{(1)}$) theories. This remark is based on the fact that as $L \to \infty$ in the $K=1$ series, the SU(N) Gross–Neveu models are recovered. This implies that the models for finite L are actually restrictions of what we call a "deformed" Gross–Neveu model. This is a new model; by "deformed" we refer to the idea that reintroducing a coupling constant breaks the G symmetry to the quantum group symmetry $\mathcal{U}_q(G)$.

In order to exhibit the generality of the construction we have included results from some forthcoming publications by two of us [17, 18].

2. Perturbed coset models and fractional spin currents

In this section, we review some basic facts concerning the construction and the properties of the coset models. We also show how it is possible to choose a relevant perturbation such that part of the underlying algebraic structure is preserved.

As stated in the introduction, we will only be concerned with the coset models $G_K \otimes G_L / G_{K+L}$. The embedding of G_{K+L} in $G_K \otimes G_L$ is the diagonal embedding. Algebraically, the coset models are defined through the GKO construction [20]: The Virasoro generators are the difference of the Sugawara generators. Namely, if $T_K(z)$ denotes the Sugawara stress tensor for representations of $G^{(1)}$ at level K, the stress tensor of the coset model is $T(z) = T_K(z) + T_L(z) - T_{K+L}(z)$. Its central charge is

$$c\left(\frac{G_K \otimes G_L}{G_{K+L}}\right) = c(G_K) + c(G_L) - c(G_{K+L})$$

$$= \text{rank}(G) - \frac{12K|\rho|^2}{(K+L+h^*)(L+h^*)} + \text{rank}(G)\left(\frac{Kh - h^*}{K+h^*}\right), \quad (2.1)$$

with ρ the Weyl vector of G; $12|\rho|^2 = \text{rank}(G)h^*(h+1)$. Here $c(G_K)$ is the central charge of the Sugawara operators: $c(G_K) = K \dim G/(K+h^*)$; $h(h^*)$ are the Coxeter (dual) number of G ($\dim(G) = (h+1)\text{rank}(G)$). In eq. (2.1), we decomposed the coset central charge in a way which reveals the existence of a Feigin–Fuchs-like construction: the first two terms in eq. (2.1) represent the central charge of a Feigin–Fuchs field valued in the Cartan subalgebra of G, whereas the second term is the central charge of the parafermions of $G_K/[U(1)]^{\text{rank}(G)}$ [21, 22].

The Hilbert spaces of the coset models are made of the branching spaces of the GKO construction. We denote the latter by $[(K;\Lambda)\otimes(L;\Lambda')/(K+L;\Lambda'')]$ where Λ, Λ' and Λ'' are integrable highest weight representations of $G^{(1)}$ at the appropriate level. They are defined by the decomposition of the tensorial product of the representations $\mathscr{H}(K;\Lambda)$ and $\mathscr{H}(L;\Lambda')$ of $G^{(1)}$:

$$\mathscr{H}(K;\Lambda)\otimes\mathscr{H}(L;\Lambda')=\sum_{\Lambda''}\mathscr{H}(K+L;\Lambda'')\otimes\left[\frac{(K;\Lambda)\otimes(L;\Lambda')}{(K+L;\Lambda'')}\right]. \tag{2.2}$$

To each branching space is associated a field, also denoted by $[(K;\Lambda)\otimes(L;\Lambda')/(K+L;\Lambda'')](z)$, which is primary with respect to $T(z)$. By construction, its conformal weight $\Delta_{\Lambda''}^{\Lambda;\Lambda'}$ is given by

$$\Delta_{\Lambda''}^{\Lambda;\Lambda'}=\frac{\mathrm{Cas}(\Lambda)}{2(K+h^*)}+\frac{\mathrm{Cas}(\Lambda')}{2(L+h^*)}-\frac{\mathrm{Cas}(\Lambda'')}{2(K+L+h^*)}+n \tag{2.3}$$

with nonnegative integer n. $\mathrm{Cas}(\Lambda)$ denotes the quadratic Casimir operator in the representation Λ which is $(\Lambda,\Lambda+2\rho)$, where ρ is half the sum of positive roots of G. The integer n depends on the depth at which the highest weight $(K+L;\Lambda'')$ appears in $\mathscr{H}(K;\Lambda)\otimes\mathscr{H}(L;\Lambda')$.

Coset models can be thought of as representations of some chiral algebras [22–24]. The chiral algebras are not unique; they can be either local, e.g. Casimir or W-algebra, or nonlocal. It is the nonlocal point of view that we will use. Namely, for fixed K but variable L, we will think about the coset models as representations of the nonlocal algebra generated by the nonlocal coset field $J^{(K)}(z)$,

$$J^{(K)}(z)=\left[\frac{(K;\mathrm{Adjoint})\otimes(L;\cdot)}{(K+L;\cdot)}\right](z). \tag{2.4}$$

The dot denotes the scalar representation. Its conformal weight is

$$\Delta(J^{(K)})=1+h^*/(K+h^*). \tag{2.5}$$

Note that this field exists only for $K\geqslant 2$, because the adjoint representation is integrable if and only if the level is larger than or equal to two. Note also that we choose the scalar representation at level L and $K+L$ in order for the conformal weight to be independent of L.

By construction, a multiplet of the algebra generated by $J^{(K)}(z)$ is made of branching spaces. The multiplets are labeled by two highest weights Λ' and Λ'' at level L and $K+L$, respectively:

$$\{\Lambda', \Lambda''\} = \sum_{\Lambda} \left[\frac{(K;\Lambda) \otimes (L;\Lambda')}{(K+L;\Lambda'')} \right]. \tag{2.6}$$

In eq. (2.6), the sum over the integrable highest weights $(K;\Lambda)$ is restricted to those which belong to the equivalence class of $(\Lambda'' - \Lambda')$ in P/Q. P(Q) is the weight (root) lattice of G.

Before analyzing perturbation theory, let us point out that it is possible to generalize the construction (2.4) to other weights (in addition to the adjoint representation). The generalized construction yields new currents only for the non-simply laced algebras because the new currents are in one-to-one correspondence with equivalence classes in $Q/Q^{\vee}$, where $Q^{\vee}$ is the long root lattice of G. These currents are always generated by one current with conformal weight

$$\Delta = \frac{x_s h}{K+h^*} \quad \text{Mod integer}, \tag{2.7}$$

with x_s equal to half the length squared of the short root.

Let us now look at perturbation theory. There exist relevant perturbations of coset models such that there is a nonlocal conserved current in the perturbed theory. The nonlocal conserved current is associated to the current (2.4). The appropriate perturbing field $\Phi_{\text{pert}}(z,\bar{z})$ is "dual" to the current $J^{(K)}(z)$:

$$\Phi_{\text{pert}}(z,\bar{z}) = \left[\frac{(K;\cdot) \otimes (L;\cdot)}{(K+L;\text{Adjoint})} \right](z,\bar{z}). \tag{2.8}$$

It induces a relevant perturbation because

$$\Delta(\Phi_{\text{pert}}) = 1 - \frac{h^*}{K+L+h^*}. \tag{2.9}$$

The field $\Phi_{\text{pert}}(z,\bar{z})$ is local with respect to the current $J^{(K)}(z)$. Indeed, the fusion rules are

$$[J^{(K)}] \times [\Phi_{\text{pert}}] = [\Psi] \tag{2.10}$$

with

$$[\Psi] = \left[\frac{(K;\text{Adjoint}) \otimes (L;\cdot)}{(K+L;\text{Adjoint})} \right],$$

$$\Delta(\Psi) = \frac{Lh^*}{(K+h^*)(K+L+h^*)}, \qquad \Delta(J^{(K)}) + \Delta(\Phi_{\text{pert}}) = 2 + \Delta(\Psi). \tag{2.11}$$

Thus, the operator product expansion (OPE) (2.10) is local. We can use arguments similar to those used by Zamolodchikov in ref. [2] to obtain the equation of motion of the current $J^{(K)}(z)$ to first order in perturbation theory. Namely,

$$\frac{\partial}{\partial \bar{z}} J^{(K)}(z, \bar{z}) = \lambda \oint_z \frac{\mathrm{d}w}{2\pi i} \Phi_{\text{pert}}(w, \bar{z}) J^{(K)}(z) . \tag{2.12}$$

The residue in the OPE (2.10) being a total derivative, there is a conserved current of spin $1 + h^*/(K+h^*)$ in the perturbed theory. More precise computation gives

$$\frac{\partial}{\partial \bar{z}} J^{(K)}(z, \bar{z}) = \frac{\partial}{\partial z} H^{(K)}(z, \bar{z}) \tag{2.13}$$

with $H^{(K)}(z,\bar{z}) = -\lambda C((K+h^*)/L)\Psi(z)\Phi_{\text{pert}}(\bar{z})$, where C is the structure constant of the OPE (2.10). The global conserved charge $Q^{(K)}$

$$Q^{(K)} = \oint (\mathrm{d}z J^{(K)}(z, \bar{z}) + \mathrm{d}\bar{z} H^{(K)}(z, \bar{z})) \tag{2.14}$$

has Lorentz spin $h^*/(K+h^*)$.

Similarly, by interchanging the role of z and $\bar{z}$ we obtain a conserved current

$$\frac{\partial}{\partial z} \bar{J}^{(K)}(z, \bar{z}) = \frac{\partial}{\partial \bar{z}} \overline{H}^{(K)}(z, \bar{z}) , \tag{2.15}$$

with the global conserved charge

$$\overline{Q}^{(K)} = \oint (\mathrm{d}\bar{z} \bar{J}^{(K)}(z, \bar{z}) + \mathrm{d}z \overline{H}^{(K)}(z, \bar{z})) \tag{2.16}$$

of Lorentz spin $-h^*/(K+h^*)$.

The field by which we are perturbing is symmetric in K and L. Therefore by exchanging the role played by K and L we can construct conserved charges $Q^{(L)}$ and $\overline{Q}^{(L)}$ with Lorentz spin $\pm h^*/(L+h^*)$. It was these symmetries that were constructed for the RSG in ref. [6]. In the approach we are developing here one set of conserved charges, say $Q^{(K)}$, will be associated to an internal symmetry algebra acting on the multiplet of fundamental Toda (or FF) fields. For example in the SU(2) cosets, if $K=2$ the symmetry is supersymmetry, and for general K the symmetry is the fractional supersymmetry constructed over the quantum SU(2) algebra with $q = \exp(i\pi h^*/(K+h^*))$. As we will see, the exact S-matrices are invariant under both the $Q^{(L)}$ and $Q^{(K)}$ symmetries.

The two charges $Q^{(K)}$ and $Q^{(L)}$ commute (or anticommute depending on the Klein factor we choose):

$$[Q^{(K)}, Q^{(L)}] = 0 . \tag{2.17}$$

In the unperturbed CFT this follows from the OPE between the currents $J^{(K)}$ and $J^{(L)}$:

$$[J^{(K)}]\times[J^{(L)}]=[\Psi'], \tag{2.18}$$

with

$$\Psi'=\left[\frac{(K;\mathrm{Adj})\otimes(L;\mathrm{Adj})}{(K+L;\cdot)}\right],\qquad \Delta(J^{(K)})+\Delta(J^{(L)})=2+\Delta(\Psi'). \tag{2.19}$$

One can easily check that the commutation relation (2.17) also holds to first order in perturbation theory. One can also similarly prove

$$\left[Q^{(K)},\overline{Q}^{(L)}\right]=0.$$

The above analysis does not demonstrate that the perturbed models are integrable. In our general framework, integrability is established by relating these models to restrictions of integrable soliton equations. This will be done for some specific cases in the sequel.

Finally let us comment on the renormalization group (RG) flows. From what is already known for the simplest coset models [13, 25–28], it is often conjectured that, in one direction (say $\lambda<0$) the perturbation yields a massive theory (i.e. in the IR limit, the theory is trivial), whereas in the other direction (say $\lambda>0$) the theory is a massless theory (i.e. in the IR limit the theory is another CFT). More precisely, in the latter case, it is conjectured that for $L\gg K$, the RG flow defined by the perturbation (2.8) maps the coset models $G_K\otimes G_L/G_{K+L}$ into the coset models $G_K\otimes G_{L-K}/G_L$. In other words the RG flow shifts $L\to L-K$. Moreover, the UV field $\Phi_{\mathrm{pert}}(z,\bar{z})$, eq. (2.8), flows into the IR dual field $J(z,\bar{z})$, eq. (2.4) with $(L\to L-K)$.

3. Soliton spectrum and S-matrices for the SU(2) cosets

In this section we consider the perturbations of the SU(2) cosets proposed in the last section for arbitrary K and L.

Let us presume that the perturbations define integrable models. The perturbing operators (2.8) are invariant under the duality transformation $K\leftrightarrow L$. Thus we require that the S matrix respects this duality. We also require that when K or L equals 1, we recover the known result for RSG. In the last section we have seen that the S matrix should be invariant under two independent symmetries $Q^{(K)}$ and $Q^{(L)}$. This fact supports the idea that the S-matrix should be the tensor product of two factors, where each factor is separately invariant under one of the symmetries. All of these requirements taken together lead to a unique conjecture for the S matrix.

As usual, we parametrize the energy and momentum of the asymptotic particles in terms of rapidity θ_i:

$$P_i = m_i \sinh\theta_i, \qquad E_i = m_i \cosh\theta_i. \tag{3.1}$$

For two particle scattering, define $\theta = \theta_1 - \theta_2$. We propose the following two-particle S-matrix:

$$S^{(K,L)}(\theta) = S^{(K)}_{\mathrm{RSG}}(\theta) \otimes S^{(L)}_{\mathrm{RSG}}(\theta). \tag{3.2}$$

Above, each of the factors $S^{(K)}_{\mathrm{RSG}}$ or $S^{(L)}_{\mathrm{RSG}}$ is a restricted sine-Gordon S matrix, and is described in detail in ref. [6]. More precisely, for the arbitrary K, L the spectrum consists of kinks:

$$K_{ab;a'b'}(\theta); \qquad b = a \pm \tfrac{1}{2}, \qquad b' = a' \pm \tfrac{1}{2},$$
$$a, b \in \left\{0, \tfrac{1}{2}, 1, \dots j^{(K)}_{\max} = K/2\right\}, \qquad a', b' \in \left\{0, \tfrac{1}{2}, 1, \dots j^{(L)}_{\max} = L/2\right\}. \tag{3.3}$$

An asymptotic N-particle state can be described as

$$\left| K_{a_0a_1;a'_0a'_1}(\theta_1) K_{a_1a_2;a'_1a'_2}(\theta_2) \dots K_{a_{N-1}a_N;a'_{N-1}a'_N}(\theta_N) \right\rangle_{\mathrm{in,out}}. \tag{3.4}$$

The S matrix for the process

$$K_{da;d'a'}(\theta_1) + K_{ab;a'b'}(\theta_2) \to K_{dc;d'c'}(\theta_2) + K_{cb;c'b'}(\theta_1) \tag{3.5}$$

is given by the matrix elements

$$S^{(K)\,ab}_{\mathrm{RSG}dc}(\theta) \cdot S^{(L)\,a'b'}_{\mathrm{RSG}d'c'}(\theta). \tag{3.6}$$

The factors $S^{(K)}_{\mathrm{RSG}}$ are proportional to [6]

$$S^{(K)\,ab}_{\mathrm{RSG}dc} \propto \sinh\left(\frac{\theta}{K+2}\right)\left(\frac{[2a+1][2c+1]}{[2d+1][2b+1]}\right)^{1/2} \delta_{db} + \sinh\left(\frac{i\pi - \theta}{K+2}\right)\delta_{ac}, \tag{3.7}$$

where

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad \text{with} \quad q = -\exp\left(\frac{-i\pi}{K+2}\right). \tag{3.8}$$

It is easy to see that eq. (3.2) satisfies all of our requirements. It is manifestly dual. When K (resp. L) equals 1, $S^{(K)}_{\mathrm{RSG}}$ (resp. $S^{(L)}_{\mathrm{RSG}}$) becomes trivial, i.e. equals the identity, since it corresponds to the S matrix of a massive Majorana fermion. Furthermore, $S^{(K)}_{\mathrm{RSG}}$ (resp. $S^{(L)}_{\mathrm{RSG}}$) is invariant under a $Q^{(K)}$ (resp. $Q^{(L)}$) fractional supersymmetry; this follows from the on-shell construction in ref. [6] for RSG★.

★ Bazhanov and Reshetikhin have independently found the above tensor product form of the S-matrix for some apparently related spin chains on the lattice (private communication).

The S matrix (3.2) has some interesting consequences for some other quantum field theories, namely, perturbations of SU(2) WZW models, and for super sine-Gordon (SSG) theory and its fractional super relatives. These connections will be explored in the next sections.

4. Perturbations of WZW models

Consider the perturbations of the coset $\mathrm{G}_K \otimes \mathrm{G}_L/\mathrm{G}_{K+L}$ proposed in sect. 2 in the limit that one of the levels, say L, goes to infinity. The CFT becomes the WZW model G_K. The perturbing operators in this limit have dimension $(1,1)$ and can be identified with the Kac–Moody currents $J^a(z)$. Thus in this limit, the perturbed CFT has the action

$$S = S^{\mathrm{WZW}} + \frac{\lambda}{2\pi i}\int \mathrm{d}^2 z \sum_a J^a(z)\bar{J}^a(\bar{z})\,. \tag{4.1}$$

The fractional supersymmetry $Q^{(K)}$ survives in this limit. In the WZW theory the $Q^{(K)}$ symmetry is generated by the current

$$J^{(K)}(z) = q_{ab} J^a_{-1} \Phi^b(z)\,, \tag{4.2}$$

where q_{ab} is the Killing form, J^a_{-1} is a mode of the Kac–Moody current ($J^a(z) = \sum_n J^a_n z^{-n-1}$), and $\Phi^b(z)$ is the chiral primary field in the adjoint representation of G. The $Q^{(L)}$ symmetry has Lorentz spin 0 and becomes identified with an internal symmetry G.

The above model (4.1) is closely related to the principal chiral model (PCM) with Wess–Zumino term [29]. The action for such a model can be taken as

$$S = S^{\mathrm{WZW}} + \alpha \int \mathrm{d}^2 z\, \mathrm{Tr}\big(\partial_\mu g^{-1} \partial^\mu g\big)\,, \tag{4.3}$$

where $g(z,\bar{z})$ is taken to be an element of the group G. The second term in eq. (4.3) is the action for the PCM, and it has the same form as the kinetic term in S^{WZW}. The manner in which the models (4.1) and (4.3) differ can be seen by the identification of the Kac–Moody currents in terms of the field g:

$$J^a(z) t^a = -\frac{K}{2}\partial_z g g^{-1}\,, \qquad \bar{J}^a(\bar{z}) t^a = -\frac{K}{2} g^{-1}\partial_{\bar{z}} g\,, \tag{4.4}$$

where t^a are matrices generating G [30]. The perturbation in (4.1) is thus

$$\big(\lambda K^2/8\pi i\big)\int \mathrm{d}^2 z\, \mathrm{Tr}\big(\partial_z g g^{-1}\big)\big(g^{-1}\partial_{\bar{z}} g\big)\,. \tag{4.5}$$

We turn now to what happens to the spectrum in the limit $L \to \infty$. We specialize the discussion to the case G = SU(2). This question can be studied by examining the S matrix factor $S_{\mathrm{RSG}}^{(L)}$ in this limit. Recall that in the derivation of $S_{\mathrm{RSG}}^{(L)}$ from a restriction of sine-Gordon theory, the restriction came about by decomposing the multisoliton Hilbert space into irreducible representations of the quantum group $\mathscr{U}_q(\mathrm{sl}(2))$, and removing states in multiplets with SU(2)-spin less than $j_{\max} = L/2$. In the limit L goes to infinity, $j_{\max}$ goes to infinity and the restriction is thus undone. This implies that the new spectrum is given by

$$K_{ab}^{\pm}(\theta), \qquad a, b \in \{0, \tfrac{1}{2}, 1, \dots, \tfrac{1}{2}K\}. \tag{4.6}$$

The extra quantum numbers $\pm$ refer to the original two-dimensional vector space of the sine-Gordon soliton quantum numbers. The S matrix for these particles is then the $L \to \infty$ limit of

$$S(\theta) = S_{\mathrm{RSG}}^{(K)}(\theta) \otimes S_{\mathrm{SG}}\left(x = \mathrm{e}^{\theta/(L+2)}, q = -\mathrm{e}^{-i\pi/(L+2)}\right), \tag{4.7}$$

where S_{SG} is the 4×4 S-matrix of sine-Gordon (SG) solitons. (See ref. [6] for the conventions we are following in describing the SG S-matrix in terms of the variables x and q.)

The $L \to \infty$ limit in eq. (4.7) is somewhat delicate. Define $\epsilon = 1/(L+2)$, and let

$$x = e^{\epsilon\theta} \approx 1 + \epsilon\theta, \qquad q = -\mathrm{e}^{-i\pi\epsilon} \approx -1 + i\pi\epsilon. \tag{4.8}$$

We now take the $\epsilon \to 0$ limit. Recall that the SG S-matrix [31] can be written as

$$S_{\mathrm{SG}}(x, q) = u(x, q)\sigma\mathscr{R}(x, q)\sigma^{-1}, \tag{4.9}$$

[6], where σ is a gauge transformation,

$$P\mathscr{R}(x, q) = x\hat{R}^{+} - x^{-1}\hat{R}^{-} \tag{4.10}$$

$$\hat{R}^{+} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \qquad \hat{R}^{-} = (\hat{R}^{+})^{-1}, \tag{4.11}$$

and $u(x, q)$ is an overall scalar factor required for unitarity,

$$u(x, q) = \frac{1}{(1 - x^2 q^2)} \left[\prod_{j=1}^{\infty} \frac{(1 - x^2 q^{4-4j})(1 - x^{-2} q^{2-4j})}{(1 - x^{-2} q^{4-4j})(1 - x^2 q^{2-4j})} \right]. \tag{4.12}$$

In eq. (4.10), P is a permutation operator. In the limit $\epsilon \to 0$,

$$\hat{\mathscr{R}}(x,q) \approx 2\epsilon \begin{pmatrix} i\pi-\theta & 0 & 0 & 0 \\ 0 & i\pi & \theta & 0 \\ 0 & \theta & i\pi & 0 \\ 0 & 0 & 0 & i\pi-\theta \end{pmatrix}$$

$$\equiv 2\epsilon \hat{\mathscr{R}}^{\mathrm{rat}}(\theta). \tag{4.13}$$

In addition we have

$$2\epsilon u(x,q) \approx \frac{1}{(i\pi-\theta)} \left[\prod_{j=1}^{\infty} \frac{[i\pi(2j-2)+\theta]}{[i\pi(2j-2)-\theta]} \frac{[i\pi(2j-1)-\theta]}{[i\pi(2j-1)+\theta]} \right] \equiv v(\theta). \tag{4.14}$$

The gauge transformations σ can be disregarded in the limit. Thus the S matrix for the particles $K^{\pm}_{ab}(\theta)$ is given by

$$S(\theta) = S^{(K)}_{\mathrm{RSG}}(\theta) \otimes S^{\mathrm{rat}}(\theta), \tag{4.15}$$

where

$$S^{\mathrm{rat}}(\theta) = v(\theta) P \hat{\mathscr{R}}^{\mathrm{rat}}(\theta). \tag{4.16}$$

The factor $P\hat{\mathscr{R}}^{\mathrm{rat}}(\theta)$ is a so-called rational solution of the Yang–Baxter (YB) equation. (The terminology is this: trigonometric solutions of the YB equation involve the functions $\exp(\alpha\theta)$, whereas the rational solutions are polynomial in θ.) This rational factor was anticipated due to the fact that the $Q^{(L)}$ symmetry has Lorentz spin 0 in this limit and corresponds to an internal symmetry G, which is carried by the factor $S^{\mathrm{rat}}(\theta)$. What was unexpected is the RSOS factor $S^{(K)}_{\mathrm{RSG}}$ in eq. (4.15). This factor makes manifest the hidden $Q^{(K)}$ fractional supersymmetry of the current–current perturbation of the WZW models. Reshetikhin has informed us of some very interesting recent work on the solution of higher SU(2) spin-chains; he finds that the exact Bethe ansatz methods reveal the same RSOS factor [32].

Finally, let us consider the double limit K and $L \to \infty$. We have already taken the $L \to \infty$ limit and shown that the resulting model is a perturbation of the SU(2) level-K WZW model. As $K \to \infty$, the action (4.1) is dominated by the kinetic term for the WZW field g, and is thus nothing other than the PCM. This can be seen by rescaling the current $J \to J/\sqrt{K}$, then taking the $K \to \infty$ limit. Just as for the $L \to \infty$ limit of the $Q^{(L)}$ symmetry discussed above, the $Q^{(K)}$ symmetry becomes an internal symmetry G in this limit. Thus the full resulting symmetry is a $\mathrm{G} \otimes \mathrm{G}$ symmetry, characteristic of the PCM. Following the reasoning above, the $K \to \infty$ limit of eq. (4.15) yields an S matrix that is the tensor product of two rational

S-matrices, and thus manifests the $G \otimes G$ symmetry. This agrees with the result found in ref. [33].

5. Minimal superconformal series and (fractional) super sine-Gordon theory

In this section we illustrate the connection to integrable soliton equations by considering the case of $SU(2)_K \otimes SU(2)_L/SU(2)_{K+L}$ for $K=2$ and variable L. This series of CFTs constitute the superminimal series [34, 35]. As we will see the perturbed models are restrictions of the supersymmetric extension of sine-Gordon (RSSG) theory. We also present the S matrix for the ordinary SSG theory and for its fractional super generalizations.

5.1. PERTURBATION OF SUPERSYMMETRIC FF CONSTRUCTION AND SSG THEORY

From the viewpoint of the general formalism of sect. 2, the superpartner to the energy–momentum tensor is the current $J(z)$ in eq. (2.4), with dimension $\frac{3}{2}$. Thus if we perturb the models with the operator Φ_{pert} given in eq. (2.8), the supersymmetry will not be broken. This operator has dimension

$$\Delta(\Phi_{\text{pert}}) = (L+2)/(L+4) \tag{5.1}$$

by eq. (2.9). The integrability of the above perturbed models can be established by relating them to the SSG theory at special rational values of the coupling.

The fields of the supersymmetric Feigin–Fuchs (FF) construction for the superminimal series [35] can be related to the SSG fields. Introduce the usual superspace coordinates z, θ (and $\bar{z}, \bar{\theta}$) and covariant derivative

$$D = \partial_\theta + \theta\partial_z \quad \text{and} \quad \bar{D} = \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}\,. \tag{5.2}$$

As usual $z = \frac{1}{2}(t+ix)$. The holomorphic fields of the FF construction consist of a single superfield $\Phi' = \phi' + \theta\psi'$, with the propagator

$$\Phi'(z_1,\theta_1)\Phi'(z_2,\theta_2) = -\log(z_1 - z_2 - \theta_1\theta_2)\,. \tag{5.3}$$

The energy–momentum tensor is

$$T(z) = \tfrac{1}{2}\partial\phi'\partial\phi' - \tfrac{1}{2}\psi'\partial\psi' - i\sqrt{2}\,\alpha_0\partial^2\phi'\,, \tag{5.4}$$

where

$$\alpha_0 = \frac{1}{\sqrt{2(L+2)(L+4)}}\,. \tag{5.5}$$

Define the super vertex operators

$$V_\alpha(z,\theta) = \exp\big(i\sqrt{2}\,\alpha\Phi'(z,\theta)\big). \tag{5.6}$$

The screening operators are

$$V_\pm = \int \mathrm{d}z\, \mathrm{d}\theta\, V_{\alpha\pm}(z,\theta) \tag{5.7}$$

and have dimension 1, where

$$\alpha_+ = \sqrt{\frac{L+4}{2(L+2)}}\,, \qquad \alpha_- = -\sqrt{\frac{L+2}{2(L+4)}}\,.$$

The primary fields of the minimal series are

$$\Phi_{m,n} = \exp\big(i\sqrt{2}\,\alpha_{m,n}\Phi'\big), \tag{5.8}$$

where

$$\alpha_{m,n} = \tfrac{1}{2}(1-m)\alpha_+ + \tfrac{1}{2}(1-n)\alpha_- \tag{5.9}$$

with $1 \leqslant m \leqslant L+1$, $1 \leqslant n \leqslant L+3$. The Neveu–Schwarz sector is given by $n-m=$ even; the Ramond sector by $n-m=$ odd.

Consider now the SSG theory with the euclidean action

$$S = (1/\beta^2)\int \mathrm{d}^2z\, \mathrm{d}^2\theta\big[D\Phi\,\overline{D}\Phi + m\cos\Phi\big]. \tag{5.10}$$

The constant β is a coupling constant. We take the convention $\mathrm{d}^2z = i\,\mathrm{d}x\,\mathrm{d}t/2$. Following the reasoning in ref. [6] the super FF fields are identified with the SSG fields by requiring one of the operators in the potential of eq. (5.10) to be a screening operator. Expanding $\cos\Phi = \frac{1}{2}[\exp(i\Phi) + \exp(-i\Phi)]$, we take the $\exp(-i\Phi)$ term to be a screening operator. Since the SSG propagator is

$$\Phi(z_1,\theta_1)\Phi(z_2,\theta_2) \sim -(\beta^2/4\pi)\log(z_1 - z_2 - \theta_1\theta_2), \tag{5.11}$$

the SSG fields are related to the FF fields by the rescaling

$$\Phi = \big(\beta/\sqrt{4\pi}\big)\Phi'. \tag{5.12}$$

Thus we identify

$$-i\frac{\beta}{\sqrt{4\pi}}=i\sqrt{2}\,\alpha_{-}\quad\Rightarrow\quad\frac{\beta^{2}}{8\pi}=\frac{L+2}{2(L+4)}\,. \tag{5.13}$$

This identification has also been made in ref. [36].

Since the screening operator has dimension 1, the part of the action that includes only the free piece and the screening operator can be considered as a CFT; it is a super Liouville theory. The extra term in the action $\exp(i\beta\Phi'/\sqrt{4\pi})$ is treated as a perturbation and is equivalent to the $\Phi_{1,3}$ primary field as can be seen from eq. (5.8). The dimension of this operator is $(L+2)/(L+4)$, in agreement with the general result (2.9). It must be emphasized that the decomposition of the action (5.10) into a conformal piece and perturbation is partly heuristic. The action for the conformal piece is not sufficient to encode the truncation of the Hilbert space one performs in the super FF construction (projection of null vectors). Thus the spectrum of the perturbed super minimal series is not equivalent to the spectrum of the SSG, but must be obtained as a restriction of it, as will be described below. Also, the ordinary (unrestricted) SSG theory does not have a background charge and corresponds to $c=\frac{3}{2}$ in the massless limit.

The above relation between perturbed super FF theory and SSG can be further justified using perturbation theory, as was done for the SG theory in ref. [8]. As for eq. (2.12), if a CFT is perturbed by an operator of the form $(\lambda/2\pi i)\int \mathrm{d}^{2}z\ \Phi_{\mathrm{pert}}(z,\bar{z})$, then the equations of motion become

$$\frac{\partial}{\partial\bar{z}}F(z,\bar{z})=\lambda\oint_{z}\frac{\mathrm{d}w}{2\pi i}\Phi_{\mathrm{pert}}(w,\bar{z})F(z)\,. \tag{5.14}$$

In order to preserve the $\mathbb{Z}_2$ symmetry $\phi'\to-\phi'$ of the super FF construction, we make the $\phi_{1,3}$ perturbation $\mathbb{Z}_2$ invariant by taking it to be

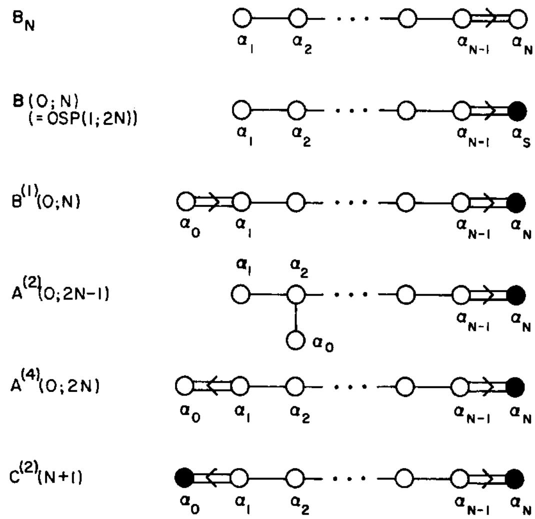
$$\Phi_{\mathrm{pert}}=-\frac{m}{2}\cos(\phi)\psi(z)\bar{\psi}(\bar{z})\,, \tag{5.15}$$

where m is considered as a perturbation parameter. (For the discussion of equations of motion we have set $\beta/\sqrt{4\pi}=1$.) The equations of motion computed from eqs. (5.14) and (5.15) are

$$\partial_{\bar{z}}\psi=-\frac{m}{2}\cos(\phi)\bar{\psi}\,,\qquad\partial_{z}\bar{\psi}=\frac{m}{2}\cos(\phi)\psi$$

$$\partial_{z}\partial_{\bar{z}}\phi=\frac{m}{2}\psi\bar{\psi}\sin\phi-\left[\frac{m^{2}}{4}\cos\phi\sin\phi\right], \tag{5.16}$$

TABLE 1
Dynkin diagrams for (affine) super Lie algebras



*The convention followed for these Dynkin diagrams is that the darkened circles refer to the fermionic simple roots.

except for the last term in brackets. The above equations (including the term in brackets) are the same as follow from the action (5.10); thus to first order in perturbation theory, the super FF fields satisfy SSG equations of motion. The additional second order term in the third equation of (5.16) arises from elimination of the auxiliary field F in $\Phi = \phi + \theta\psi + \bar{\theta}\bar{\psi} + \theta\bar{\theta}F$. This extra term can be recovered in perturbation theory by requiring Φ_{pert} to be invariant under supersymmetry.

The SSG theory is integrable because it is equivalent to Toda theory on the twisted super affine Lie algebra $C^{(2)}(2)$ [37–39]. The equation of motion of the SSG theory can be written as a super zero-curvature condition. The generalized Dynkin diagram of $C^{(2)}(2)$ can be found in table 1. The convention followed in these diagrams is that the darkened roots refer to the fermionic ones. Let $(e_i, f_i, h_i \equiv \alpha_i^\vee,\ i = 0, 1)$ be a Chevalley basis for $C^{(2)}(2)$ satisfying

$$[h_i, h_j] = 0, \qquad \{e_i, f_j\} = \delta_{ij} h_j,$$

$$[h_i, e_j] = (\alpha_i^\vee, \alpha_j) e_j, \qquad [h_i, f_j] = -(\alpha_i^\vee, \alpha_j) f_j.$$

The generators can be written as

$$e_1 = G_+^{(1)}, \qquad f_1 = G_-^{(1)}, \qquad h_1 = -H/2$$
$$e_0 = \lambda G_-^{(2)}, \qquad f_0 = (1/\lambda) G_+^{(2)}, \qquad h_0 = H/2, \tag{5.17}$$

where λ is the affine, or spectral, parameter. The generators in eq. (5.17) satisfy

$$\begin{aligned}
&\left[H, G_\pm^{(1,2)}\right] = \pm\tfrac{1}{2} G_\pm^{(1,2)},\\
&\left\{G_+^{(1)}, G_-^{(1)}\right\} = -\left\{G_+^{(2)}, G_-^{(2)}\right\} = -\tfrac{1}{2}H,\\
&\left\{G_\pm^{(1)}, G_\pm^{(1)}\right\} = \left\{G_\pm^{(2)}, G_\pm^{(2)}\right\} = \pm\tfrac{1}{2}J^\pm,\\
&\quad [J^+, J^-] = 2H, \qquad [H, J^\pm] = \pm J^\pm,\\
&\left\{G_\pm^{(1)}, G_\pm^{(2)}\right\} = 0,\\
&\left\{G_+^{(1)}, G_-^{(2)}\right\} = \left\{G_-^{(1)}, G_+^{(2)}\right\} = \frac{1}{2i}\tilde{H}.
\end{aligned}$$

Note that H and $G_\pm^{(1)}$ themselves generate OSp(1, 2). The lowest-dimensional representation of $C^{(2)}(2)$ is given by

$$H = \tfrac{1}{2}\,\mathrm{diag}(1, 0, -1), \qquad \tilde{H} = \tfrac{1}{2}\,\mathrm{diag}(1, 2, 1),$$

$$G_+^{(1)} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}, \qquad G_-^{(1)} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0\\ -1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix},$$

$$G_+^{(2)} = \frac{1}{2i}\begin{pmatrix} 0 & -1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}, \qquad G_-^{(2)} = \frac{1}{2i}\begin{pmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}. \tag{5.18}$$

Define a superfield valued in the Cartan subalgebra $\Phi = \Phi(x, \theta, \bar{\theta})H$. The SSG equations of motion are equivalent to the super zero-curvature conditions

$$\{D + A, \bar{D} + \bar{A}\} = 0, \tag{5.19}$$

where

$$D + A = e^{-i\Phi} D\, e^{i\Phi} + i\sqrt{m}\; e^{i\Phi} \Lambda\, e^{-i\Phi}$$
$$\bar{D} + \bar{A} = e^{i\Phi} \bar{D}\, e^{-i\Phi} + i\sqrt{m}\; e^{-i\Phi} \bar{\Lambda}\, e^{i\Phi}. \tag{5.20}$$

Above, Λ and $\bar{\Lambda}$ are sums of the positive and negative simple roots respectively, i.e.

$$\Lambda = e_0 + e_1, \qquad \bar{\Lambda} = f_0 + f_1. \tag{5.21}$$

The zero-curvature formulation along with its associated quantum inverse scattering problem can in principle be used to solve the model exactly. However as we will see, for our purposes there is no need to carry this analysis through.

The SSG theory has the conserved topological current,

$$J^{\mu} = (1/\pi)\epsilon^{\mu\nu}\partial_{\nu}\phi. \tag{5.22}$$

The soliton sector consists of field configurations with non-zero topological charge. Consider a solution with $\psi = \bar{\psi} = 0$. In this situation the equations of motion (5.16) are the same as for sine-Gordon theory up to a factor of 2 (since $\sin 2\phi = 2\sin\phi\cos\phi$). Using known topological solutions of sine-Gordon theory, a solution of the SSG theory with nonzero topological charge is

$$\phi_{\rm sol}(x) = 2\tan^{-1}[\exp(mx/2)]. \tag{5.23}$$

The topological current (5.22) is normalized such that the above solution has topological charge $+1$. Note that the normalization of the topological current (5.22) differs from the sine-Gordon normalization by a factor of 2, a fact that will be important later. A general soliton solution has a fermionic partner of the same mass by supersymmetry. There also exist antisoliton solutions with topological charge -1, which also have fermionic partners. Thus we expect that the soliton sector consists of two supersymmetry doublets with opposite topological charge; all particles have the same mass.

5.2. SOLITON SPECTRUM AND S MATRICES OF THE RESTRICTED SSG THEORY

Our conjecture is that the soliton spectrum and S matrices of the above perturbations of the superminimal series are as given in eqs. (3.2) and (3.3). Namely,

$$S_{\rm RSSG}(\theta) = S_{\rm RSG}^{(K=2)}(\theta) \otimes S_{\rm RSG}^{(L)}(\theta). \tag{5.24}$$

In order to develop the RSSG theory in analogy with RSG, we would need to start with the S matrix of the soliton sector of SSG, and from it derive the RSSG result. Unfortunately, despite several attempts, the S matrix of SSG solitons remains unknown. However, since we already are in possession of the soliton spectrum and S matrix for the restricted model, we can reverse the logic and undo the restriction. This will provide a new solution of the SSG theory, and will be described in the next section.

Here we will discuss some of the features of the RSSG S-matrix, relating them to the super FF and super Landau–Ginsburg constructions. The restriction of quantum numbers in the RSSG S-matrix can be compared to the implicit restriction in the super FF construction, following the reasoning in ref. [6]. The topological charge t of an operator is defined as $[\int dx J^0, \mathscr{O}] = t\mathscr{O}$, where J^μ is the topological current. From the form of the primary fields $\Phi_{m,n}$ in eq. (5.8), the relation between the SSG and super FF fields (5.12), and the properly normalized topological current (5.22), we find that the primary fields $\Phi_{m,n}$ have topological charge

$$t_{m,n} = \frac{4\beta}{\sqrt{8\pi}} \alpha_{m,n} = (1-m) - (1-n)\left(\frac{L+2}{L+4}\right). \tag{5.25}$$

The fields with integral topological charge are $\Phi_{m,1}$. Since $m \leqslant L+1$, the minimum integral topological charge is $-L$. Thus there are effectively $L+1$ minima in the restricted SSG potential, in accordance with the super Landau–Ginsburg description [40]. Compare this to the spectrum of kinks $K_{ab;a'b'}(\theta)$. Consider the quantum numbers $a', b' \in \{0, \frac{1}{2}, 1, \ldots, j_{\max}^{(L)}\}$. If we interpret the labels a', b' as describing a kink that connects two degenerate vacua, then the number of such minima is $2j_{\max}^{(L)} + 1 = L+1$, in agreement with the above reasoning. The other quantum numbers a, b label the states of a supermultiplet, which consists of two particles, $K_{0\frac{1}{2};a'b'}$ and $K_{\frac{1}{2}1;a'b'}$.

Finally we point out that the RSSG theory at the special value of $L=2$ is an integrable perturbation of an $N=2$ CFT at $c=1$. This follows from the fact that both the $Q^{(K)}$ and $Q^{(L)}$ symmetries are ordinary supersymmetries in this case.

5.3. THE (FRACTIONAL) SSG S-MATRIX

We present here the result for the S matrix of the SSG solitons. As explained above, we can undo the restriction of the RSSG theory to deduce the SSG S-matrix. In this procedure the factor $S_{\mathrm{RSG}}^{(K)}$ is unaffected; however the factor $S_{\mathrm{RSG}}^{(L)}$ becomes an ordinary sine-Gordon soliton S matrix, with a different dependence on the coupling. Since $\beta^2/4\pi = (L+2)/(L+4)$, we are led to define a function γ of the coupling β as

$$L + 2 \equiv \gamma(\beta) = \frac{\beta^2/2\pi}{1-\beta^2/4\pi}. \tag{5.26}$$

The SSG spectrum then consists of kinks $K_{ab}^{\pm}(\theta)$, $a, b \in \{0, \frac{1}{2}, 1\}$, with S matrix

$$S^{\mathrm{SSG}}(\theta) = S_{\mathrm{RSG}}^{(K=2)}(\theta) \otimes S_{\mathrm{SG}}\left(x = e^{\theta/\gamma}, q = -e^{-i\pi/\gamma}\right). \tag{5.27}$$

The supersymmetry of this S matrix is carried by the $S^{(2)}_{\rm RSG}$ factor and acts on the a, b indices. The on-shell supersymmetry leaves the second index b of the kink $K^{\pm}_{ab}$ unchanged. Thus the superpartners are $K^{\pm}_{0\frac{1}{2}}$ and $K^{\pm}_{\frac{1}{2}1}$. For $K=2$ the fractional charges Q and $\overline{Q}$ satisfy the supersymmetry algebra [13]

$$Q^2 = P, \qquad \overline{Q}^2 = \overline{P}, \qquad Q\overline{Q} + \overline{Q}Q = T, \tag{5.28}$$

where T is a topological charge. The additional topological generators in the above algebra are characteristic of supersymmetry algebras in soliton sectors [41]. Thus we have the unexpected result that the S matrix of SSG solitons is equivalent to the tensor product of an S matrix for SG solitons with the RSOS S-matrix for perturbations of the tricritical Ising model. The complete spectrum, including bound states, can be deduced as usual from the pole structure of the above S matrix.

Note further that as $\beta^2/4\pi \to 1$, this is the same as the $L \to \infty$ limit of analysis of sect. 4. This means that the perturbations of the level-2 WZW models considered there are equivalent to SSG theory at $\beta^2 = 4\pi$. Another way of thinking about this phenomenon is that in the $L \to \infty$ limit of the coset theories, global G invariance is achieved. However, reintroducing the coupling β^2 breaks G to the quantum group $\mathscr{U}_q$(G). The enhanced SU(2) symmetry is exactly analogous to what occurs in the SG theory at $\beta^2/8\pi = 1$ [42].

The above analysis can be extended for arbitrary K. The result yields a spectrum $K^{\pm}_{ab}$, $a, b \in \{0, \frac{1}{2}, \dots, \frac{1}{2}K\}$. The resulting S matrix is for a new integrable model we will refer to as the fractional super sine-Gordon theory (FSSG). Due to the generalized FF construction of the coset models, the field content can be taken to be a single boson plus a $\mathbb{Z}_K$ parafermion. We define the coupling β via the propagator normalization as in (5.11). This model has the S matrix

$$S^{(K)}_{\rm FSSG}(\theta) = S^{(K)}_{\rm RSG}(\theta) \otimes S_{\rm SG}\left(x = \mathrm{e}^{\theta/\gamma},\, q = -\mathrm{e}^{-i\pi/\gamma}\right), \tag{5.29}$$

where now

$$\gamma(\beta) = \frac{K^2\beta^2}{8\pi - K\beta^2}.$$

The above function $\gamma(\beta)$ was computed in exactly the same manner as for (5.26), namely by using results from the generalized FF construction in ref. [22]. Also, as for the SG and SSG theories, for special values of the coupling, namely

$$\beta^2/8\pi = 1/K,$$

these theories are equivalent to the perturbations of the level-K WZW models studied in sect. 4. These theories will be more fully developed elsewhere [18].

6. Perturbations of WB-series and Toda on classical affine superalgebras

In this section, we develop the connection between WB models and Toda models on superalgebras. It is the first step towards the generalization of the previous section on SU(2) to larger algebras. The main point we want to stress is how one should add fermions and/or parafermions when moving to nonsimply laced algebras and/or higher levels. In other words, one should consider Toda-like models on super algebras or more exotic algebras. The WB models are the coset models $B_N^{[1]} \otimes B_N^{[L]}/B_N^{[L+1]}$. (The numbers in the square brackets are the levels of the representations of $B_N^{(1)}$.) The Virasoro central charge of WB_N-models at level L is

$$c = N\left[1 - \frac{(2N-1)(2N+1)}{(2N+L)(2N+L-1)}\right] + \frac{1}{2}. \tag{6.1}$$

The parafermion of the level-one representations of $B_N^{(1)}$ is a free fermion (because B_N has only one simple short root whose square length is one). Therefore the Feigin–Fuchs-like construction of the WB_N models involves bosonic fields $\partial\boldsymbol{\phi}$ valued in the Cartan subalgebra of B_N together with a free fermion ψ [14–16]. The $\frac{1}{2}$ term in (6.1) is the central charge of the fermion and the full Feigin–Fuchs stress tensor is

$$T(z) = -\tfrac{1}{2}(\partial\boldsymbol{\phi})^2 + i\alpha_0 \boldsymbol{\rho}\cdot\partial^2\boldsymbol{\phi} + \tfrac{1}{2}\psi\partial\psi \tag{6.2}$$

with $\boldsymbol{\rho}$ the Weyl vector of B_N, and

$$\alpha_0 = \frac{1}{\sqrt{(L+2N-1)(L+2N)}}.$$

The screening operators are in two-to-one correspondence with the simple roots, $\alpha_1, \dots, \alpha_N$, of B_N. They are

$$V_\pm^k(z) = \exp(i\alpha_\pm \boldsymbol{\alpha}_k \cdot \boldsymbol{\phi}), \quad \text{for } k = 1, \dots, N-1,$$
$$V_\pm^N(z) = \psi(z)\exp(i\alpha_\pm \boldsymbol{\alpha}_N \cdot \boldsymbol{\phi}) \tag{6.3}$$

with $\alpha_+ + \alpha_- = \alpha_0$, and $\alpha_+\alpha_- = -1$. Our convention for the simple roots is given in table 1. The screening operators have conformal weight one.

The WB_N models were initially introduced as representations of a Casimir vertex operator algebra of B_N, i.e. the local chiral algebra of currents $W_{(j)}(z)$ with conformal weight $\Delta(j) = m(j) + 1$ where $m(j)$ runs over the exponents of B_N: $m(j) = 1, 3, \dots, 2N-1$. But as we explained in sect. 2, the coset models can also be interpreted as representations of a nonlocal algebra. However since the case we are considering is $K=1$, the current (2.4) does not exist, and we have to use the alternative construction mentioned in sect. 2. For the algebra B_N, the only nontrivial representative of $\mathrm{Q}/\mathrm{Q}^\vee$ is the highest weight ω_V of the vector representation. The nonlocal current associated to ω_V is

$$G(z)=\left[\frac{\underline{(1;\square)}\otimes(L;\cdot)}{(1+L;\cdot)}\right](z),\qquad(\square=\omega_\mathrm{V}). \tag{6.4}$$

Its conformal weight is $\Delta(\mathrm{G})=N+\frac{1}{2}$ (one-half is the conformal weight of $(1;\square)$ and N is the depth of $(1+L;\cdot)$ in the tensorial product $(1;\square)\otimes(L;\cdot)$). Obviously for $N=1$, the current $G(z)$ is the supercurrent and we are simply describing the super Feigin–Fuchs construction for $\mathrm{G}=\mathrm{SU}(2)$. Moreover because $\mathrm{SO}(3)\cong\mathrm{SU}(2)/\mathbb{Z}_2$, the coset $\mathrm{B}_1^{[1]}\otimes\mathrm{B}_1^{[L]}/\mathrm{B}_1^{[L+1]}$ is equivalent to $\mathrm{A}_1^{[2]}\otimes\mathrm{A}_1^{[2L]}/\mathrm{A}_1^{[2L+2]}$.

Toda models on the finite-dimensional simply-laced Lie algebra X were shown to be equivalent to the WX-models [23, 24]. However in the case of a nonsimply laced algebra the quantization used in ref. [23] encounters difficulties due to screening operators. In the case of the algebra B_N, there is only one short simple root, α_N (of length one). One finds that a free fermion should be added to the vertex operators associated to the short simple root. Thus the short simple root acquires a fermionic character, and, starting from the Lie algebra B_N we are led to the super Lie algebra $\mathrm{B}(0;N)=\mathrm{OSp}(1,2N)$. It is the only classical super Lie algebra [43] (with non-singular Cartan matrix). Its Dynkin diagram is shown in table 1.

Let us now describe the $\mathrm{B}(0;N)$ Toda models. Let $(e_j,f_j,\alpha_j^\vee)$, $j=1$ to $N-1$, be the bosonic generators of $\mathrm{B}(0;N)$ and $(e_s,f_s,\alpha_s^\vee)$, $s=N$, the fermionic ones. Let $\Phi(z,\theta)=\phi(z)+\theta\psi(z)+\overline{\theta\psi}(z)+\theta\bar{\theta}F(z)$ be a superfield valued in the Cartan subalgebra of $\mathrm{B}(0;N)$:

$$\Phi(z,\theta)=\sum_j\alpha_j^\vee\Phi^j(z,\theta)+\sum_s\alpha_s^\vee\Phi^s(z,\theta). \tag{6.5}$$

Then the $\mathrm{B}(0;N)$ Toda equations of motion for the superfield $\Phi(z,\theta)$ are

$$\overline{D}D\Phi+\frac{m}{\beta}\sum_s\alpha_s^\vee\,\mathrm{e}^{\beta(\Phi,\alpha_s)}+\frac{m^2}{\beta}\theta\bar{\theta}\sum_j\alpha_j^\vee\,\mathrm{e}^{\beta(\Phi,\alpha_j)}=0, \tag{6.6}$$

where β is a coupling constant; $\alpha_j(\alpha_s)$ are the bosonic (fermionic) roots of $\mathrm{B}(0;N)$: $\alpha_j=\epsilon_j-\epsilon_{j+1}$, for $j=1$ to $N-1$; $\alpha_s=\epsilon_N$ for $s=N$.

Note that the $\theta\bar{\theta}$ term associated to the bosonic simple roots breaks the supersymmetry. Only $\mathrm{OSp}(1,2)$ has no bosonic simple roots; in that case (6.6) reduces to the supersymmetric Liouville equation in agreement with the equivalence between the WB_1 models and the minimal superconformal models, as explained in sect. 5.

As in the super Liouville case, eq. (6.6) can be written as a super zero-curvature condition. Namely, if we define

$$D+A=\mathrm{e}^{-\beta\Phi/2}D\,\mathrm{e}^{\beta\Phi/2}+\sqrt{m}\;\mathrm{e}^{\beta\Phi/2}\Lambda\,\mathrm{e}^{-\beta\Phi/2},$$

$$\bar{D}+\bar{A}=\mathrm{e}^{\beta\Phi/2}\bar{D}\,\mathrm{e}^{-\beta\Phi/2}+\sqrt{m}\;\mathrm{e}^{-\beta\Phi/2}\bar{\Lambda}\,\mathrm{e}^{\beta\Phi/2} \tag{6.7}$$

with $\Lambda=\Sigma_s e_s+\theta\Sigma_j e_j$ and $\bar{\Lambda}=\Sigma_s f_s+\bar{\theta}\Sigma_j f_i$, then the super zero-curvature condition

$$\{D+A,\bar{D}+\bar{A}\}=0 \tag{6.8}$$

is equivalent to the B(0; N) Toda equations. The super zero-curvature condition (6.8) ensures the classical integrability of the theory.

In components, eq. (6.6) tells us that the fermions of the superfields $\Phi^j(z,\theta)$ associated to the bosonic roots α_j decouple. They can be consistently set to zero. The other equations of motion are

$$\partial\bar{\partial}\,\phi^j(z)=(m^2/\beta)\mathrm{e}^{\beta(\phi,\alpha_j)} \tag{6.9}$$

for the bosonic simple roots, and

$$\partial\bar{\psi}=\frac{m}{\beta}\psi\,\mathrm{e}^{\beta(\phi,\alpha_N)},\qquad \bar{\partial}\psi=-\frac{m}{\beta}\bar{\psi}\,\mathrm{e}^{\beta(\phi,\alpha_N)},$$

$$\partial\bar{\partial}\,\phi^N=-\frac{m}{\beta}\psi\bar{\psi}\,\mathrm{e}^{\beta(\phi,\alpha_s)}-\frac{m^2}{\beta^2}\,\mathrm{e}^{2\beta(\phi,\alpha_N)} \tag{6.10}$$

for the fermionic simple root α_N. (We have set $\psi^N\equiv\psi$.)

The quantization of the (super) Liouville models by Gervais and Neveu [44] leads to the minimal (super) conformal series [45,46]. In the same way, quantization of eqs. (6.9) and (6.10) leads to the Feigin–Fuchs construction of the WB_N models; the field contents are the same: a bosonic field valued in the Cartan subalgebra of B_N plus one fermion associated to the short simple root of B_N; the improved stress tensor of the B(0; N) Toda model is the stress tensor (6.2); and for specific values of the coupling constant $\beta(\beta=i\alpha_-)$ the vertex operators in the r.h.s. of the eqs. (6.9) and (6.10) become screening operators.

Let us now look at perturbed WB theories by comparing them to Toda models on affine superalgebras. More precisely we will analyze which of the relevant perturbations of the WB models can be described by restricted Toda models on affine superalgebras★. We will only be concerned with the classical affine superalgebras (with nonsingular Cartan matrix). They are named $\mathrm{B}^{(1)}(0;N)$, $\mathrm{A}^{(2)}(0;2N-1)$, $\mathrm{A}^{(4)}(0;2N)$ and $\mathrm{C}^{(2)}(N+1)$ [43]. The corresponding Dynkin diagrams are

★ We expect that these models are not strictly restrictions of an affine super Toda theory but actually of a related model with extra interactions for the non-simple roots, just as for perturbations of the simply-laced cosets (see sect. 7).

given in table 1. Note that in all cases the horizontal subalgebra of the affine algebra (the one obtained by deleting the extended roots α_0 in the Dynkin diagram) is the superalgebra OSp$(1,2N)$. Moreover, in all cases except $C^{(2)}(N+1)$, the extended root is a bosonic root.

The equations of motion for the Toda models on affine superalgebras have the same form as eq. (6.6) for Toda models on finite superalgebras. However in the affine case, the sum is over all the simple roots of the affine superalgebra. Thus one has to project eq. (6.6) onto the horizontal algebra taking into account that the extended root is $\alpha_0 = d - \theta$ with d the derivation and $\theta^{\vee} = \Sigma_j a_j^{\vee}\alpha_j^{\vee} + \Sigma_s a_s^{\vee}\alpha_s^{\vee}$ where $a^{\vee}$ are the dual Kăc labels of the affine superalgebra. The projected equations of motion can be easily written down. In the limiting case $C^{(2)}(2)$, it gives the equations of motion of the SSG model studied in the previous section.

As in the finite case, the equations of motion of the Toda models on affine superalgebra can be written as a zero curvature condition. The connection is the one given in eq. (6.7) but it is now written with the generators e and f of the loop superalgebra.

Because in all cases the horizontal algebra of the affine superalgebra is the superalgebra OSp$(1,2N)$, all Toda models on affine superalgebra can be interpreted as perturbations of the OSp$(1,2N)$ Toda models. By looking at the equation of motion it is easy to determine which perturbing fields the affine Toda models correspond to. But not all the perturbations are consistent with the restriction. A case by case inspection reveals that:

(i) The $B^{(1)}(0;N)$ Toda models describe a perturbation by an irrelevant field.

(ii) The restricted $A^{(2)}(0;2N-1)$ Toda models describe the perturbation of the WB models by the relevant perturbing field $\Phi_{\text{pert}} = [(1;\cdot)\otimes(L;\cdot)/(1+L;\text{Adjoint})]$. Its conformal weight is $\Delta_{\text{pert}} = (L+1)/(L+2N)$. As for simply laced algebras [10] it is conjectured that the S matrix of this perturbed WB model with $L=1$ is the minimal S-matrix of the $A^{(2)}(0;2N-1)$ Toda models. The spectrum of the masses of the bosons depends only on the Dynkin diagram irrespective of the fermionic or bosonic character of the simple roots [39]. Therefore the masses of the bosons of the $A^{(2)}(0;2N-1)$ Toda models are the same as the masses of bosons of the $B^{(1)}(N)$ Toda model, namely,

$$M_k = 2M\sin(\pi k/2N), \quad \text{for } k=1,\dots,N-1,$$

$$M_N = M. \tag{6.11}$$

Also the WSO(N) models at level one (i.e. $SO(n)_1 \otimes SO(n)_1/SO(n)_2$) are equivalent to a free boson compactified on a circle of radius $R = \sqrt{n}/2$. The perturbations we are describing correspond to the perturbation of the Gaussian models by the vertex operators of weight $(1/2R^2)$.

(iii) The $A^{(4)}(0;2N)$ Toda models have no restriction for integrable perturbation of WB models.

(iv) The number of fermionic fields in the $C^{(2)}(N+1)$, $N \geq 2$, Toda model is two, therefore there is no obvious interpretation of the $C^{(2)}(N+1)$ Toda as perturbation of WB models. But if there exists a restriction strong enough to identify the two fermions, then the restricted $C^{(2)}(N+1)$ Toda models will be interpretable as a perturbation by the relevant field $[(1;\cdot)\otimes(L;\cdot)/(L+1;\square)]$ with conformal weight $\Delta=(L+N)/(L+2N)$.

We stress once more that as in the case of the simply laced algebras the connection between the perturbed WB models and the minimal Toda models on superalgebras is expected to hold only for $L=1$. For $L>1$ the perturbed WB models are related to generalizations of the Toda models.

7. Other groups: conjectured S matrices of the SU(N) cosets

To show how the S matrices of the SU(2) cosets generalize for larger algebras, let us give the conjectured S matrices for the SU(N) cosets: $SU(N)_K \otimes SU(N)_L/SU(N)_{K+L}$. The generalization to other groups will be clear. Because of the invariance under the two nonlocal conserved charges $Q^{(K)}$ and $Q^{(L)}$, the S matrices factorize into a tensorial product of two RSOS-like S matrices multiplied by some CDD factors. To be more precise let us first introduce the highest weights ω_n, $n=1,\dots,r=N-1$, of the fundamental representations R_n of SU(N). Note that the weights ω_n define integrable representations of $SU(N)^{(1)}$ at level one. In the perturbed SU(N) cosets there are r families of kinks, which we denote by (K_n), $n=1,\dots,r$. For fixed n, all the kinks of the family (K_n) have the same mass M_n:

$$M_n = M\frac{\sin(n\pi/N)}{\sin(\pi/N)}\,; \qquad n=1,\dots,N-1. \tag{7.1}$$

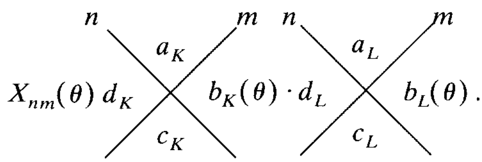
More generally, for the group G the mass spectrum is equivalent to the spectrum of masses in the Toda ($G^{(1)}$) theory.

The kinks in (K_n) are labeled by four highest weights of SU(N). A kink of rapidity θ is $K_n\binom{a_K \to b_K}{a_L \to b_L}(\theta)$ where a_K, b_K (resp. a_L, b_L) denote highest-weight representations of $SU(N)^{(1)}$ at level K (resp. L). A pair of weights $(a \to b)$ is said to be n-admissible iff the representation b appears in the tensorial product $a \otimes R_n$. Note that the multiplicity of a representation in $(a \otimes R_n)$ is always one so that we do not have to specify the SU(N) homomorphism defining the decomposition of $(a \otimes R_n)$ into b. Only n-admissible pairs of $(a \to b)$ appear in the kinks of the family (K_n). The kinks $K_n\binom{a_K \to b_K}{a_L \to b_L}$ can be thought of as the tensorial product of two elementary kinks of the perturbed coset theory with one of the levels equal to 1; or alternatively of an SU(N) RSOS statistical model.

The S matrix for the kinks in the family (K_n) and (K_m) is conjectured to be

$$S_{nm}(\theta) = X_{nm}(\theta) S^{[K]}_{nm}(\theta) \otimes S^{[L]}_{nm}(\theta) , \tag{7.2}$$

where $S^{[K]}_{nm}(\theta)$ is (up to a scalar function) the trigonometric RSOS $(\omega_n; \omega_m)$ $\mathscr{R}$-matrix with the quantum group parameter $q = -\exp(-i\pi/(K+h^*))$, where $h^* = N$. See e.g. ref. [47] for a precise definition. Following the diagrammatic notation of ref. [6] the matrix elements of $S_{nm}(\theta)$ for two kink scattering are



(7.3)

$X_{nm}(\theta)$ are the standard SU(N) CDD factors; see e.g. refs. [10, 12, 33, 49].

Eq. (7.2) is the most natural generalization of the SU(2) cosets S-matrices. Instead of giving a detailed proof of it (the details will be described elsewhere), we will just offer three checks:

(i) For K (resp. L) = 1, the RSOS S-matrix factors $S^{[K]}$ (resp. $S^{[L]}$) are trivial, i.e. are equal to 1. Thus when $K = L = 1$, the S matrices S_{nm} reduce to the CDD factors $X_{nm}(\theta)$. This gives the known result [10–12, 49]. In order to clarify a confusion in the existing literature, we point out that the resulting $r = N - 1$ particles are certainly not the particles corresponding to the r bosonic fields of the SU(N) affine Toda theory. One should not be misled by the fact that the spectrum of masses of the affine Toda fields is the same as the spectrum for the families of kinks. For this special case the S-matrix can be alternatively derived simply from this spectrum of masses and the bootstrap. Furthermore, as we will indicate below, it is not even correct to identify these $K = L = 1$ models as restrictions of SU(N) affine Toda theory.

(ii) We can consider the theories in the limits considered in sect. 4 for SU(2). For $K = 1$, $L \to \infty$, the models we are discussing become the SU(N) WZW models at level $K = 1$ perturbed by the $J^a(z)\bar{J}^a(\bar{z})$ operators. These are nothing but the SU(N) Gross–Neveu models. This can be easily established by bosonization of the Gross–Neveu models. (See e.g. ref. [48] and references therein for a study of the Gross–Neveu models.) On the other hand when $L \to \infty$ the SU(N) trigonometric $\mathscr{R}$ matrices become the rational SU(N) $\mathscr{R}$ matrices. Therefore the S matrices (7.2) go into the known S matrices of the SU(N) Gross–Neveu models. This result indicates that the perturbations of the cosets $G_1 \otimes G_L/G_{1+L}$ can be formulated as restrictions of a "deformed" Gross–Neveu model, rather than a restriction of affine Toda theory. By "deformed" we refer to the fact that by reintroducing a coupling β into the perturbed models, the symmetry G is broken to the quantum group $\mathscr{U}_q$(G), in complete analogy to the results in sect. 4. These new models will be further described elsewhere [19]. We present here (7.4) the action for these models for the case of SU(N):

$$S = \frac{1}{\beta^2}\int d^2z\left[\partial_z\phi\cdot\partial_{\bar{z}}\phi + m^2\sum_{\alpha>0}\cos\left(\frac{\alpha\cdot\phi}{\sqrt{2}}\right)\right], \tag{7.4}$$

where α is a positive root of SU(N); ϕ is a field valued in the Cartan subalgebra and β is a coupling constant. Note that this action is hermitian, unlike the action for the affine Toda theory. The above action (7.4) can be derived at the SU(N) invariant point by inserting the bosonized form of the Kac–Moody currents (see the vertex operator constructions at level 1 in refs. [14–16] into the action (4.1). Following the reasoning above for SU(2), the S-matrices for the SU(N) deformed Gross–Neveu models are conjectured to be

$$S(\theta)_{nm} = X_{nm}\hat{S}_{nm}(\theta),$$

where X_{nm} are the standard CDD factors used above, and $\hat{S}_{nm}$ are proportional to (up to scalar factors required for unitarity) the trigonometric vertex-type $\mathscr{R}$-matrices of the quantum group SU$_q$(N) acting in the tensor product ($\omega_n\otimes\omega_m$). In other words this S-matrix is the unrestricted form of eq. (7.2) for one of the levels K or L equal to 1.

(iii) Following the reasoning in sect. 4 for the SU(2) case, when $K\to\infty$, $L\to\infty$, the perturbed SU(N) coset models become the SU(N) principle chiral models. On the other hand, as explained above, the trigonometric SU(N) $\mathscr{R}$-matrices go into the rational SU(N) $\mathscr{R}$-matrices. Therefore in that limit, the S matrices (7.2) become a tensorial product of two SU(N) rational $\mathscr{R}$-matrices times the CDD factors. These are the S matrices of the SU(N) nonlinear sigma models found in ref. [33].

8. Conclusions

We have identified the main features of a classification of integrable massive quantum field theory that parallels the classification of rational CFT. We can summarize our strategy as follows. We consider the generalized FF fields as the field content of an integrable field theory and its (fractional) super generalizations. Then we restrict this soliton theory to obtain a minimal series. The general scheme, along with connections to other models, is better summarized by the diagram in fig. 1. The simplest realization of this pattern is for the $K=1$ series which involve deformed Gross–Neveu models. Having provided the structure of the perturbed coset models, we may now proceed to "unrestrict" these models as we have done for some specific cases above. In general one thereby deduces the exact soliton spectrum and S matrices of some new integrable QFTs. We will defer the complete formulation to a future publication [19].

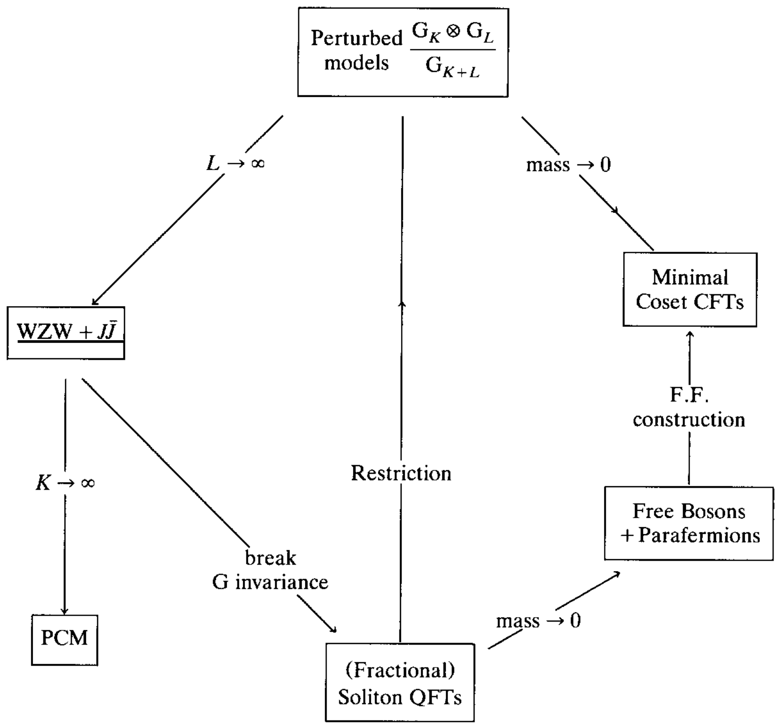


Fig. 1. Integrable massive QFTs and their relations.

Our results may be of interest for applications to condensed matter theory. We have indirectly demonstrated the existence of fractional Lorentz-spin excitations in a variety of hamiltonian spin systems in one spatial dimension. For example, the fractional supersymmetries we found in the current–current perturbations of the WZW models can be taken as evidence that the higher su(2)-spin Heisenberg chains do in fact have fractional Lorentz-spin quasi-particles.

An interesting question is whether the fractional supersymmetries have a classical analog. We do not have a definite answer to this question. However, in the appendix we give an example of such a classical symmetry by constructing a fractional superspace.

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Note added in proof

N. Reshetikhin has informed us that he and Bazhanov intend to publish their work on apparently related spin chains (referred to above) in the near future.

Appendix A

CLASSICAL FRACTIONAL SUPERSYMMETRY

The formulation of supersymmetric field theories is simplified by the introduction of a superspace. Remarkably, there exists a generalization to a fractional superspace. We will present this formalism here, deferring a discussion of its utility to a future publication. For the RSG theories at level L, define $M=(L+2)/2$. It was argued [6] that the fractional supersymmetries satisfy the algebra

$$Q^M = P^+ , \qquad \overline{Q}^M = P^- , \tag{A.1}$$

where $P^{\pm}$ are light-cone components of momentum. Introduce variables $z, \theta, \bar{z}, \bar{\theta}$, and a complex parameter q satisfying

$$\theta^M = \bar{\theta}^M = 0 , \qquad q^M = 1 . \tag{A.2}$$

We define a derivative ∂_θ such that

$$\partial_\theta \cdot \theta = \partial_\theta(\theta) + q\theta\partial_\theta , \tag{A.3}$$

with $\partial_\theta(\theta) = 1$. This derivative has the desirable property that $\partial_\theta(\theta^M) = 0$. In general

$$\partial_\theta(\theta^n) = \frac{1-q^n}{1-q}\theta^{n-1} . \tag{A.4}$$

The derivative is not the unique one satisfying $\partial_\theta(\theta^M) = 0$. There are in fact $M-1$ of them. Here we will only need one other, denoted δ_θ, satisfying

$$\delta_\theta \cdot \theta = \delta_\theta(\theta) + q^{-1}\theta\delta_\theta . \tag{A.5}$$

We find

$$\delta_\theta(\theta^n) = \frac{1-q^{-n}}{1-q^{-1}}\theta^{n-1} .$$

It is not difficult to prove that

$$\delta_\theta\partial_\theta - q\partial_\theta\delta_\theta = 0 . \tag{A.6}$$

One can find a representation of the algebra (A.1) on the fractional superspace:

$$Q = \partial_\theta + a\theta^{M-1}\partial_z, \qquad a = \left[\prod_{i=0}^{M-2} \frac{1-q^{i+1}}{1-q}\right]^{-1}. \tag{A.7}$$

Q satisfies

$$Q^M = \partial_z. \tag{A.8}$$

A covariant derivative satisfying

$$DQ - qQD = 0 \tag{A.9}$$

can be constructed using the other derivative

$$D = \delta_\theta + qa\theta^{M-1}\partial_z. \tag{A.10}$$

Define integrals as

$$\int \mathrm{d}\theta\, \theta^n = \delta_{n,M-1}.$$

An example of an action with the fractional supersymmetry is

$$S = \int \mathrm{d}^2 z\, \mathrm{d}^2\theta\, D\Phi\, \bar{D}\Phi,$$

where Φ is a fractional superfield $\Phi(x,\theta,\bar{\theta}) = \phi(x) + \theta\psi_1(x) + \theta^2\psi_2(x) + \dots$.

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