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Twisted Bethe equations from a twisted S-matrix

Changrim Ahn,[a] Zoltan Bajnok,[b] Diego Bombardelli[a,c] and Rafael I. Nepomechie[d]

[a] *Department of Physics and Institute for the Early Universe, Ewha Womans University, DaeHyun 11-1, Seoul 120-750, S. Korea*

[b] *Theoretical Physics Research Group, Hungarian Academy of Sciences, 1117 Budapest, Pázmány s. 1/A, Hungary*

[c] *Physics Department and Theoretical Physics Center, University of Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal*

[d] *Physics Department, University of Miami, P.O. Box 248046, Coral Gables, FL 33124, U.S.A.*

E-mail: ahn@ewha.ac.kr, bajnok@elte.hu, diego.bombardelli@fc.up.pt, nepomechie@physics.miami.edu

Abstract: All-loop asymptotic Bethe equations for a 3-parameter deformation of AdS_5/CFT_4 have been proposed by Beisert and Roiban. We propose a Drinfeld-Reshetikhin twist of the AdS_5/CFT_4 S-matrix, together with c-number diagonal twists of the boundary conditions, from which we derive these Bethe equations. Although the undeformed S-matrix factorizes into a product of two $su(2|2)$ factors, the deformed S-matrix cannot be so factored. Diagonalization of the corresponding transfer matrix requires a generalization of the conventional algebraic Bethe ansatz approach, which we first illustrate for the simpler case of the twisted $su(2)$ principal chiral model. We also demonstrate that the transfer matrix is spectrally equivalent to a transfer matrix which is constructed using instead untwisted S-matrices and boundary conditions with operatorial twists.

Keywords: Lattice Integrable Models, AdS-CFT Correspondence, Exact S-Matrix, Bethe Ansatz

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Contents

1 Introduction

One of the major triumphs of theoretical physics in this century has been the discovery and exploitation of integrability in AdS/CFT. (For recent reviews, see for example [1–3].) This integrability is remarkably robust. In particular, it seems to persist for the so-called β-deformed AdS_5/CFT_4 theory [4]–[17], and even for a 3-parameter deformation [6]–[9]. However, much remains to be understood about the integrability of the deformed theory, and we have sought to make progress toward that end.

We focus here on the problem of deriving the Bethe equations. For the undeformed theory, the all-loop asymptotic Bethe equations [18] have been derived [20, 21, 27] from the AdS_5/CFT_4 S-matrix [19]–[26]. For the 3-parameter deformed theory, corresponding all-loop asymptotic Bethe equations were proposed by Beisert and Roiban in [9]. A long outstanding question has been whether it is possible to derive these deformed Bethe equations from some deformed S-matrix.[1] Here we answer this question in the affirmative: the deformed Bethe equations *can* be derived from a certain Drinfeld-Reshetikhin twist [28–31] of the AdS_5/CFT_4 S-matrix, together with c-number diagonal twists of the boundary conditions.

A key point is that, although the undeformed S-matrix factorizes into a product of two $su(2|2)$ factors, the deformed S-matrix cannot be so factored. Indeed, the twist matrix connects the two $su(2|2)$ factors, and cannot be factorized into a product of separate twist matrices for the two $su(2|2)$ factors. To our knowledge, such "non-factoring" twists have not been considered previously in the literature, and it is not obvious how to diagonalize the corresponding transfer matrix. Indeed, since the transfer matrix no longer splits into a product of commuting left and right pieces, one would naively expect that such a twist leads to very complicated Bethe equations. Hence, before addressing the problem of actual interest, we first consider the simpler case of the $su(2)$ principal chiral model with a non-factoring twist. We develop techniques for this model which we subsequently use to solve the twisted AdS_5/CFT_4 problem.

The outline of this paper is as follows. In section 2 we present a heuristic argument to infer the specific non-factoring Drinfeld-Reshetikhin twist of the $su(2|2)^2$ S-matrix and twisted boundary conditions which should lead to the twisted Bethe equations in [9]. In section 3 we consider the $su(2)$ principal chiral model with a similar non-factoring twist. We develop an algebraic Bethe ansatz method for diagonalizing the transfer matrix and deriving the Bethe equations. In section 4 we consider two copies of the Hubbard model with a non-factoring twist, which is technically very similar to the twisted AdS_5/CFT_4 problem. Using the method of the previous section together with the algebraic Bethe ansatz for the Hubbard model developed by Martins and Ramos [32], we diagonalize the transfer matrix and derive the corresponding Bethe equations. Finally, in section 5, we consider the problem of actual interest; namely, the AdS_5/CFT_4 S-matrix with a non-factoring twist. We obtain the eigenvalues of the transfer matrix from the preceding results, and write down the corresponding Bethe equations. In section 6 we show that these Bethe equations

[1] Recently these equations were obtained from a twisted transfer matrix solution of the Y-system [16] corresponding to operatorial twisted boundary conditions [17].

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agree with those proposed in [9]. We close in section 7 with a brief discussion of our results. Some technical details are treated in appendices A and B. We show in appendix C that the proposed twisted S-matrix and twisted boundary conditions reproduce the wrapping correction not only for the Konishi operator [15] but also for generic multiparticle states both in the $su(2)$ and $sl(2)$ sectors analyzed in [16, 17]. We demonstrate in appendix D that the transfer matrix is spectrally equivalent to a transfer matrix which is constructed using instead *untwisted* S-matrices and boundary conditions with *operatorial* twists. It is the latter type of transfer matrix which is considered in [17]. Finally, in appendix E we transform our twisted Bethe ansatz results from the "$su(2)$" grading to the "$sl(2)$" grading.

2 Deformation from a $psu(2,2|4)$ perspective

The first indication of the equivalence between type-IIB string theory on an $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ supersymmetric four-dimensional Yang-Mills theory was their common global symmetry: namely, $psu(2,2|4)$. On the string-theory side it is the supersymmetric extension of the isometries of the background geometry, while on the gauge-theory side it is the model's superconformal invariance. This common symmetry algebra enables one to compare observables: both the energy levels of the string states and the anomalous dimensions of gauge-invariant operators are organized in the same $psu(2,2|4)$ multiplet. Recent studies suggest that the complete solution of the spectral problem can be formulated in terms of the Y-system of $psu(2,2|4)$, too [33]. Unfortunately, however, there has not been much progress yet in solving directly the Y-system beyond the leading weak-coupling order [34], or outside of the string semiclassical domain [35].

Alternatively, quantization based on the light-cone gauge has proved to be successful so far: In solving the string σ-model one chooses a generalized light-cone gauge, which turns the model into a massive integrable quantum field theory in a finite volume (prescribed by the light-cone momentum), where excitations satisfy the level-matching condition. In the infinite volume limit these excitations scatter via a factorizing scattering matrix, which can be uniquely determined from the remaining $psu(2,2|4) \to su(2|2) \otimes su(2|2)$ global symmetry together with crossing symmetry [19]–[26]. The resulting S-matrix can be used for any value of the coupling, and defines the theory completely: The full particle spectrum can be read off from its singularity structure, it governs the finite-size corrections to the energies, and via the Thermodynamic Bethe Ansatz it describes the complete spectrum for any finite volume. Unfortunately, the $psu(2,2|4)$ symmetry is broken in this description by the light-cone gauge and is realized only implicitly in the spectrum. The analogue of this phenomenon can be found on the gauge-theory side: In calculating the anomalous dimension of an operator, a BPS "vacuum" state is chosen $\mathrm{tr}(Z^L)$ which breaks the superconformal $psu(2,2|4)$ symmetry down to $su(2|2) \otimes su(2|2)$. The broken symmetry then controls the scattering of the "excitations" $\mathrm{tr}(Z^{L-k-2}\chi_1 Z^k \chi_2)$ over the background, and determines their scattering matrix. The boundary condition is provided again by the physical meaning of the trace: namely, the total momentum has to vanish. Similarly to the string case, the $psu(2,2|4)$ symmetry is not manifested in the scattering matrix, but rather in the structure of the one-loop Bethe ansatz and implicitly in the anomalous dimensions of the fields.

Integrable deformations of the $psu(2,2|4)$ structures appear both on the string-theory and on the gauge-theory sides. In both cases, the $su(4)$ part of the symmetry is Drinfeld-Reshetikhin twisted [28–31] by three parameters/charges corresponding to the Cartan generators. On the gauge-theory side, the authors of [9] present the deformed $psu(2,2|4)$ one-loop Bethe ansatz equations and conjecture the all-loop generalizations. These Bethe ansätze can be equivalently described by the asymptotical solution of a twisted Y-system, which presumably originates from the twisted $psu(2,2|4)$ symmetry.

As the S-matrix approach has turned out to be very powerful in the undeformed case, we pursue it for the deformed case too. For this we must understand how a twist of the broken $psu(2,2|4)$ symmetry shows up at the unbroken $su(2|2)\otimes su(2|2)$ level. To this end, we first analyze a toy model: an $su(4)$-invariant spin chain. There we identify two effects: a twist of the scattering matrix of the excitations, and a twist of the boundary conditions. Implementing the analogous twist at the $psu(2,2|4)$ level, we can infer the form of the twisted AdS/CFT scattering matrix together with the twisted boundary conditions. In sections 5, 6 and appendix C we test our proposal against the Bethe equations of [9] and the Lüscher corrections.

2.1 Twisted $su(4)$ spin chain

We first recall how to break the $su(4)$ symmetry down to $su(3)$ by choosing a pseudovacuum, and how the $su(3)$-invariant scattering matrix appears in this context. We then turn to the twisted problem.

Suppose we would like to solve the spectral problem for an $su(4)$ spin chain with N sites. It is defined in terms of the $su(4)$-invariant S-matrix

$$\mathbb{S}(u) = u\mathbb{I}\otimes\mathbb{I} + i\mathcal{P}\,, \tag{2.1}$$

where $\mathbb{I}$ is the 4-dimensional identity matrix, and $\mathcal{P}$ is the 16×16 permutation matrix. We are interested in the eigenvalues of the transfer matrix

$$\mathfrak{t}(u) = \mathrm{tr}_a(\mathbb{T}_a(u)) = \mathrm{tr}_a\left(\prod_{j=1}^{N}\mathbb{S}_{aj}(u)\right). \tag{2.2}$$

These eigenvalues can be calculated by the nested algebraic Bethe ansatz method. Here we are interested only in the first step of the nesting. This means to choose a pseudovacuum state

$$|0\rangle = |1,\ldots,1\rangle \equiv |1^N\rangle\,, \tag{2.3}$$

and to analyze the excitations $(2,3,4)$ over this background, which are invariant only under the unbroken $su(3)$ subgroup. In keeping with this residual symmetry, we decompose the monodromy matrix as

$$\mathbb{T}(u) = \begin{pmatrix} A(u) & B_2(u) & B_3(u) & B_4(u)\\ C_2(u) & D_{22}(u) & D_{23}(u) & D_{24}(u)\\ C_3(u) & D_{32}(u) & D_{33}(u) & D_{34}(u)\\ C_4(u) & D_{42}(u) & D_{43}(u) & D_{44}(u)\end{pmatrix}, \tag{2.4}$$

where the C_i's together with $D_{i\neq j}$ annihilate the pseudovacuum; i.e., $C_i(u)|0\rangle = D_{i\neq j}(u)|0\rangle = 0$. The diagonal elements of the monodromy matrix (which contribute to the trace) act diagonally[2]

$$A(u)|0\rangle = a(u)|0\rangle\,, \qquad D_{kk}(u)|0\rangle = \mathbb{S}_{k1}^{k1}(u)^N|0\rangle = d_k(u)|0\rangle\,, \tag{2.5}$$

and the B_i's create the three $su(3)$ excitations. A general multiparticle B-state has the form

$$B_{i_1}(v_1)\dots B_{i_K}(v_K)|0\rangle\,, \tag{2.6}$$

and we would like to diagonalize the action of A and D_{kk} on these states, as this is needed to obtain the transfer matrix eigenvalues. To do so, we need the commutation relations of the various operators, which can be obtained from the $\mathbb{STT} = \mathbb{TTS}$ relations. The B-particles are exchanged as

$$B_i(u)B_j(v) = (u-v)B_j(v)B_i(u) + iB_i(v)B_j(u) = \mathcal{S}_{ij}^{kl}(u-v)B_l(v)B_k(u)\,, \tag{2.7}$$

showing that they scatter on each other with the $su(3)$-invariant S-matrix, which we denote by $\mathcal{S}$. The action of A on the multiparticle state can be computed from

$$A(u)B_j(v) = \frac{v-u+i}{v-u}B_j(v)A(u) - \frac{i}{v-u}B_j(u)A(v)\,. \tag{2.8}$$

In computing the eigenvalue, we focus on the first "wanted" term and neglect the second "unwanted" one, as its contribution will vanish when v_i satisfy the Bethe equations. Similarly, the wanted terms resulting from acting with D_{ki} are

$$\begin{aligned} D_{ki}(u)B_j(v) &= \frac{1}{(u-v)}\left[(u-v)B_j(v)D_{ki}(u) + iB_i(v)D_{kj}(u)\right] + \dots \\ &\propto \mathcal{S}_{ij}^{lm}(u-v)B_m(v)D_{kl}(u) + \dots \end{aligned} \tag{2.9}$$

Clearly, the action of D_{ki} mixes up the $su(3)$ indices of a given state, and we have to diagonalize the following expression

$$\begin{aligned} &D_{kk}(u)B_{i_1}(v_1)\dots B_{i_K}(v_K)|0\rangle \\ &\quad\propto \mathcal{S}_{k,i_1}^{k_1 j_1}(u-v_1)\dots\mathcal{S}_{k_{K-1}i_K}^{k,j_K}(u-v_K)B_{j_1}(v_1)\dots B_{j_K}(v_K)D_{kk}(u)|0\rangle + \dots \end{aligned} \tag{2.10}$$

As $D_{k\neq i}|0\rangle = 0$, the nonvanishing elements form a trace of the reduced $su(3)$ transfer matrix

$$t(u) = \sum_{k=2}^{4}\mathcal{S}_{k,i_1}^{k_1 j_1}(u-v_1)\dots\mathcal{S}_{k_{K-1}i_K}^{k,j_K}(u-v_K)d_k(u)\,, \tag{2.11}$$

which must be diagonalized in order to finally solve the $su(4)$ eigenvalue problem. However, we shall not pursue this problem further here.

[2] We use the convention $\mathbb{S}(u)_{kl}^{mn} = u\delta_k^m\delta_l^n + i\delta_k^n\delta_l^m$.

We now would like to instead twist the $su(4)$ scattering matrix by a Drinfeld-Reshetikhin twist[3]

$$\mathbb{S} \to \tilde{\mathbb{S}} = \mathbb{F}\,\mathbb{S}\,\mathbb{F}\,, \qquad \mathbb{F} = e^{\frac{i}{2}\sum_{i,j=1}^{3}\gamma_{ij}(H_i\otimes H_j - H_j\otimes H_i)}\,, \tag{2.12}$$

where H_i are the Cartan elements of $su(4)$: $(H_i)_{kj} = \frac{1}{2}(\delta_{i,k}\delta_{i,j} - \delta_{i+1,j}\delta_{i+1,k})$. Due to the special form of the scattering matrix, only the diagonal elements are twisted

$$\tilde{\mathbb{S}}(u)^{mn}_{kl} = u\Gamma_{kl}\delta^m_k\delta^n_l + i\delta^n_k\delta^m_l\,, \tag{2.13}$$

which can be encoded in the matrix Γ. We are interested in the eigenvalues of the twisted transfer matrix

$$\tilde{\mathfrak{t}}(u) = \mathrm{tr}_a(\tilde{\mathbb{T}}_a(u)) = \mathrm{tr}_a\Big(\prod_{j=1}^{N}\tilde{\mathbb{S}}_{aj}(u)\Big)\,. \tag{2.14}$$

The pseudovacuum state $|0\rangle = |1,\ldots,1\rangle = |1^N\rangle$ is annihilated by the generators $\tilde{C}_i$ and $\tilde{D}_{i\neq j}$, and it is an eigenstate of the diagonal elements

$$\tilde{A}(u)|0\rangle = a(u)|0\rangle\,, \qquad \tilde{D}_{kk}(u)|0\rangle = \tilde{\mathbb{S}}^{k1}_{k1}(u)^N|0\rangle = \tilde{d}_k(u)|0\rangle\,. \tag{2.15}$$

The eigenvalues are now different for each k, and depend on the twist as $\tilde{d}_k(u) = (\Gamma_{k1})^N d_k(u)$. This indicates that the reduced symmetry is not even $su(3)$, which can be seen also from the way that the twisted creation operators $\tilde{B}_i$ are exchanged:

$$\tilde{B}_i(u)\tilde{B}_j(v) = (u-v)\Gamma_{ij}\tilde{B}_j(v)\tilde{B}_i(u) + i\tilde{B}_i(v)\tilde{B}_j(u) = \tilde{\mathcal{S}}^{kl}_{ij}(u-v)\tilde{B}_l(v)\tilde{B}_k(u)\,, \tag{2.16}$$

exactly with the reduced twisted scattering matrix elements. From the point of view of the reduced transfer matrix, the relevant commutation relation is twisted as follows

$$\tilde{D}_{ki}(u)\tilde{B}_j(v) \propto (\Gamma_{k1})^{-1}\tilde{\mathcal{S}}^{lm}_{ij}(u-v)\tilde{B}_m(v)\tilde{D}_{kl}(u) + \ldots \tag{2.17}$$

It will result in the twisted reduced transfer matrix

$$\begin{aligned}\tilde{t}(u) &= \sum_{k=2}^{4}\tilde{\mathcal{S}}^{k_1j_1}_{k,i_1}(u-v_1)\ldots\tilde{\mathcal{S}}^{k,j_K}_{k_{K-1}i_K}(u-v_K)(\Gamma_{k1})^{-K}\tilde{d}_k(u)\\ &= \sum_{k=2}^{4}\tilde{\mathcal{S}}^{k_1j_1}_{k,i_1}(u-v_1)\ldots\tilde{\mathcal{S}}^{k,j_K}_{k_{K-1}i_K}(u-v_K)(\Gamma_{k1})^{N-K}d_k(u)\,,\end{aligned} \tag{2.18}$$

which must still be diagonalized at the $su(3)$ level.

Focusing on the effect of the twist, we can see the emergence of two main features: (i) the reduced twisted scattering matrix $\tilde{\mathcal{S}}$ is a reduction of the twisted scattering matrix $\tilde{\mathbb{S}}$; and (ii) there is a twisted boundary condition which depends on the number of sites N and the number of particles K. The twisted boundary condition contains the twist factors Γ_{k1}, which are unphysical from the reduced-space point of view, as it is spanned only by $(2,3,4)$. Let us see how we can implement a similar twist in the AdS/CFT realm, where the full $psu(2,2|4)$ scattering matrix is not yet known.

[3] The general notion of twisting for (quasi-triangular) quasi-Hopf algebras was introduced by Drinfeld [28–30]. Reshetikhin considered [31] specific twists with elements F constructed from the Cartan generators. It is the latter type of twist which we use here. For other applications of such twists, see for example [36–39]; and for work related to quantized braided algebras, see [40–43].

2.2 Proposed twists for AdS/CFT

We would like to develop the AdS/CFT case in parallel to our $su(4)$ example. The full symmetry of the model which we wish to twist is the implicit $psu(2,2|4)$ symmetry. According to [9], one should twist the $su(4)$ R-symmetry subgroup (which corresponds to the isometries of the S^5 part of $AdS_5 \times S^5$) by its charges R_1, R_2, R_3. Let us suppose that the full $psu(2,2|4)$ scattering matrix $\mathbb{S}$ were known. Let us twist it with the charges as follows

$$\tilde{\mathbb{S}} = \mathbb{F}\,\mathbb{S}\,\mathbb{F}\,, \qquad \mathbb{F} = e^{\frac{i}{2}\sum_{i,j=1}^{3}\gamma_{ij}(R_i\otimes R_j - R_j\otimes R_i)}\,. \tag{2.19}$$

Here $\mathbb{S}$ describes how in the gauge-theory side the excitations $X_1 = X, X_2 = Y, X_3 = Z, \Psi, D, \ldots$ scatter on each other; and the action of the charges on the scalars are given by

$$R_i|X_j\rangle = \delta_{ij}|X_j\rangle\,. \tag{2.20}$$

In describing the AdS/CFT integrable model, the pseudovacuum is usually chosen to be $|0\rangle = |Z^J\rangle$. We are interested in the remaining degrees of freedom, which we consider to be excitations over this background. Clearly, in the undeformed case this choice of the vacuum breaks the $psu(2,2|4)$ symmetry down to $su(2|2)\otimes su(2|2)$, and this reduced symmetry is what labels the excitations:

$$(1,2,3,4)\otimes(\dot{1},\dot{2},\dot{3},\dot{4})\,, \tag{2.21}$$

where as usual the R-symmetry acts in the first two components. Choosing $X = 1\dot{1}$ and $Y = 2\dot{1}$, one can see that

$$R_1 = 1\otimes h + h\otimes 1\,, \quad R_2 = 1\otimes h - h\otimes 1\,, \quad R_3 = 0\,, \tag{2.22}$$

where $h = \text{diag}\left(\frac{1}{2}, -\frac{1}{2}, 0, 0\right)$.

The two effects of the twist implemented on $\mathbb{S}$ by $\mathbb{F}$ read on the reduced level as follows. First, consider how the reduced scattering matrix is twisted. Since $R_3 = 0$, the twist factors given by γ_{13} and γ_{23} have no effect, while the γ_{12} twist propagates directly through. This means that the reduction of the unknown twisted scattering matrix $\tilde{\mathbb{S}}$ must be simply the Drinfeld-Reshetikhin twist of the reduced scattering matrix,

$$\tilde{\mathcal{S}} = F\mathcal{S}F \qquad F = e^{i\gamma_{12}(R_1\otimes R_2 - R_2\otimes R_1)} = e^{2i\gamma_{12}(h\otimes 1\otimes 1\otimes h - 1\otimes h\otimes h\otimes 1)}\,, \tag{2.23}$$

where $\mathcal{S}$ is the $su(2|2)\otimes su(2|2)$-invariant AdS/CFT scattering matrix. Second, consider the twists of the boundary conditions for the particles, which come from the charge of the background and the commutation relations of the operators. Both have the form

$$e^{2i(\gamma_{13}R_1+\gamma_{23}R_2)J} = e^{2i(\gamma_{13}-\gamma_{23})(h\otimes 1)J + 2i(\gamma_{13}+\gamma_{23})(1\otimes h)J} = e^{2i(\gamma_{13}-\gamma_{23})Jh}\otimes e^{2i(\gamma_{13}+\gamma_{23})Jh}\,, \tag{2.24}$$

where J, being the R_3 charge of the background, is related to the volume of our theory.

As previewed in the Introduction, the twist matrix F in (2.23) connects the two $su(2|2)$ factors in $\mathcal{S}$, and cannot be factorized into a product of separate twist matrices for the two $su(2|2)$ factors. In order to learn how to handle such twists, we consider in the following section the analogous problem for a simpler model.

3 Twisting the $su(2)$ principal chiral model

We define the $su(2)$-invariant S-matrix $S(u)$ by

$$S(u) = S_0(u)\,[u\,\mathbb{I}\otimes\mathbb{I} + i\mathcal{P}] \tag{3.1}$$

where $\mathbb{I}$ is the 2×2 unit matrix, $\mathcal{P}$ is the 4×4 permutation matrix, and $S_0(u)$ is some scalar factor whose explicit value will not concern us here. This S-matrix acts on $V\otimes V$, where V is a 2-dimensional vector space. The S-matrix $\mathcal{S}(u)$ of the $su(2)$ principal chiral model is given by a tensor product of two copies of $S(u)$ [44, 45]. That is,

$$\mathcal{S}_{a\,\dot{a}\,b\,\dot{b}}(u) = S_{ab}(u)\,S_{\dot{a}\dot{b}}(u)\,. \tag{3.2}$$

Our convention is to arrange the four vector spaces on which $\mathcal{S}$ acts in the order $V_a\otimes V_{\dot{a}}\otimes V_b\otimes V_{\dot{b}}$. Hence,

$$\mathcal{S}_{1234}(u) = S_{13}(u)\,S_{24}(u)\,. \tag{3.3}$$

Starting from $S_{12} = S\otimes\mathbb{I}\otimes\mathbb{I}$, one can sequentially construct

$$\begin{aligned} S_{13} &= \mathcal{P}_{23}S_{12}\mathcal{P}_{23}\,, \\ S_{23} &= \mathcal{P}_{12}S_{13}\mathcal{P}_{12}\,, \\ S_{24} &= \mathcal{P}_{34}S_{23}\mathcal{P}_{34}\,, \end{aligned} \tag{3.4}$$

where $\mathcal{P}_{12} = \mathcal{P}\otimes\mathbb{I}\otimes\mathbb{I}$, etc.

In view of our proposal (2.23), we consider the Drinfeld-Reshetikhin twist

$$\tilde{\mathcal{S}}(u) = F\,\mathcal{S}(u)\,F\,, \tag{3.5}$$

where the twist matrix F is given by

$$F = e^{i\gamma_1(h\otimes\mathbb{I}\otimes\mathbb{I}\otimes h - \mathbb{I}\otimes h\otimes h\otimes\mathbb{I})}\,, \tag{3.6}$$

where γ_1 is a twist parameter, and h is the diagonal matrix

$$h = \operatorname{diag}\left(\frac{1}{2}, -\frac{1}{2}\right). \tag{3.7}$$

As already mentioned, $\mathbb{I}$ is the 2×2 unit matrix. Note that F cannot be factored between the two $su(2)$ factors.

We consider the transfer matrix[4]

$$\tilde{t}(u) = \operatorname{tr}_{a\dot{a}}\,M_{a\dot{a}}\tilde{\mathcal{T}}_{a\dot{a}}(u)\,, \tag{3.8}$$

where the monodromy matrix is constructed from the twisted S-matrix as follows

$$\tilde{\mathcal{T}}_{a\dot{a}}(u) = \tilde{\mathcal{S}}_{a\,\dot{a}\,1\,\dot{1}}(u)\cdots\tilde{\mathcal{S}}_{a\,\dot{a}\,L\,\dot{L}}(u)\,. \tag{3.9}$$

[4]Ultimately, we shall need the eigenvalues of an inhomogeneous transfer matrix, with inhomogeneities θ_j at each site j. But once one understands how to solve the homogeneous problem, it is trivial to generalize to the inhomogeneous case.

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The matrix $M_{a\dot{a}}$, which acts only in the auxiliary space and serves to twist the boundary conditions, is given by (see (2.24))

$$M = e^{i\gamma_2 h} \otimes e^{i\gamma_3 h} = \text{diag}\left(e^{i(\gamma_2+\gamma_3)/2}, e^{i(\gamma_2-\gamma_3)/2}, e^{i(\gamma_3-\gamma_2)/2}, e^{-i(\gamma_2+\gamma_3)/2}\right) , \tag{3.10}$$

where γ_2, γ_3 are additional twist parameters. The twisted S-matrix $\tilde{\mathcal{S}}(u)$ by construction [31] obeys the Yang-Baxter equation, and therefore, the twisted monodromy matrix obeys the usual intertwining relation[5]

$$\tilde{\mathcal{S}}_{a\,\dot{a}\,b\,\dot{b}}(u-v)\,\tilde{\mathcal{T}}_{a\dot{a}}(u)\,\tilde{\mathcal{T}}_{b\dot{b}}(v) = \tilde{\mathcal{T}}_{b\dot{b}}(v)\,\tilde{\mathcal{T}}_{a\dot{a}}(u)\,\tilde{\mathcal{S}}_{a\,\dot{a}\,b\,\dot{b}}(u-v)\,. \tag{3.11}$$

Also $\left[\tilde{\mathcal{S}}(u)\,, M\otimes M\right] = 0$; and therefore [46], the transfer matrix (3.8) has the commutativity property $\left[\tilde{t}(u)\,,\tilde{t}(v)\right] = 0$. The main problem is to determine the eigenvalues of the transfer matrix. Since it is not evident how to solve this problem, it is helpful to begin with the untwisted case.

3.1 Untwisted case: conventional approach

We now consider the untwisted case; i.e., $\gamma_i = 0$, and therefore both F and M are 1. In this case, the monodromy matrix is given by

$$\begin{aligned} \mathcal{T}_{a\dot{a}}(u) &= \mathcal{S}_{a\,\dot{a}\,1\,\dot{1}}(u)\cdots\mathcal{S}_{a\,\dot{a}\,L\,\dot{L}}(u) \\ &= S_{a1}(u)S_{\dot{a}\dot{1}}(u)\ldots S_{aL}(u)S_{\dot{a}\dot{L}}(u) \\ &= S_{a1}(u)\ldots S_{aL}(u)\ S_{\dot{a}\dot{1}}(u)\ldots S_{\dot{a}\dot{L}}(u) \\ &= T_a(u)\ T_{\dot{a}}(u)\,. \end{aligned} \tag{3.12}$$

Hence,

$$\mathcal{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes \begin{pmatrix} \dot{A} & \dot{B} \\ \dot{C} & \dot{D} \end{pmatrix} = \begin{pmatrix} A\dot{A} & A\dot{B} & B\dot{A} & B\dot{B} \\ A\dot{C} & A\dot{D} & B\dot{C} & B\dot{D} \\ C\dot{A} & C\dot{B} & D\dot{A} & D\dot{B} \\ C\dot{C} & C\dot{D} & D\dot{C} & D\dot{D} \end{pmatrix} \tag{3.13}$$

The transfer matrix therefore factors into a product of two commuting pieces

$$t(u) = \text{tr}_{a\dot{a}}\,\mathcal{T}_{a\dot{a}}(u) = \left[A(u)+D(u)\right]\left[\dot{A}(u)+\dot{D}(u)\right]. \tag{3.14}$$

We make the ansatz that the eigenstates of the transfer matrix are given by

$$|\Lambda\rangle = \prod_{j=1}^{m} B(u_j)|0\rangle \otimes \prod_{k=1}^{\dot{m}} \dot{B}(\dot{u}_k)|\dot{0}\rangle\,, \tag{3.15}$$

where $|0\rangle$ and $|\dot{0}\rangle$ are states with all spins up.

[5] Our case does not seem to be related to quantized braided algebras where the Yang-Baxter equation is also braided.

Evidently, the problem has factored into two copies of the XXX spin-1/2 chain, whose solution is well known. The eigenvalues $\Lambda(u)$ of the transfer matrix can therefore be easily written down,

$$\Lambda(u) = S_0(u)^{2L}\left[(u+i)^L \prod_{j=1}^{m}\left(\frac{u-u_j-i}{u-u_j}\right) + u^L \prod_{j=1}^{m}\left(\frac{u-u_j+i}{u-u_j}\right)\right] \times \left[(u+i)^L \prod_{k=1}^{\dot m}\left(\frac{u-\dot u_k-i}{u-\dot u_k}\right) + u^L \prod_{k=1}^{\dot m}\left(\frac{u-\dot u_k+i}{u-\dot u_k}\right)\right]. \tag{3.16}$$

In principle, one should derive the Bethe equations by carefully tracking the "unwanted" terms, and demanding that they cancel; however, this is a tedious computation. In practice, it is much simpler to impose the requirement that the poles of the eigenvalues should cancel.[6] In this way, we readily obtain the following Bethe equations:[7]

$$\left(\frac{u_j+i}{u_j}\right)^L = \prod_{j'\neq j}^{m} \frac{u_j-u_{j'}+i}{u_j-u_{j'}-i}\,, \qquad \left(\frac{\dot u_k+i}{\dot u_k}\right)^L = \prod_{k'\neq k}^{\dot m} \frac{\dot u_k-\dot u_{k'}+i}{\dot u_k-\dot u_{k'}-i}\,. \tag{3.17}$$

For further details, see appendix A in [45] and appendix C in [47].

3.2 Untwisted case: new approach

We have described above the "obvious" way to solve the untwisted problem. However, this approach cannot be used to solve the twisted problem, since then the transfer matrix does not factor into left and right pieces. So, now we want to solve the untwisted problem again but in a different way, *without* exploiting the factorizability of the transfer matrix. The basic idea is to develop an algebraic Bethe ansatz for the "full" monodromy matrix (3.13).

Using the well-known exchange relations between A, B, C, D, together with the result (3.13), it is not difficult to show that (see appendix A)

$$\begin{aligned}
\mathcal{T}_{11}(u)\,\mathcal{T}_{13}(v) &= \frac{u-v-i}{u-v}\mathcal{T}_{13}(v)\,\mathcal{T}_{11}(u) + \frac{i}{u-v}\mathcal{T}_{13}(u)\,\mathcal{T}_{11}(v)\,,\\
\mathcal{T}_{11}(u)\,\mathcal{T}_{12}(v) &= \frac{u-v-i}{u-v}\mathcal{T}_{12}(v)\,\mathcal{T}_{11}(u) + \frac{i}{u-v}\mathcal{T}_{12}(u)\,\mathcal{T}_{11}(v)\,,\\
\mathcal{T}_{22}(u)\,\mathcal{T}_{12}(v) &= \frac{u-v+i}{u-v}\mathcal{T}_{12}(v)\,\mathcal{T}_{22}(u) - \frac{i}{u-v}\mathcal{T}_{12}(u)\,\mathcal{T}_{22}(v)\,,\\
\mathcal{T}_{33}(u)\,\mathcal{T}_{13}(v) &= \frac{u-v+i}{u-v}\mathcal{T}_{13}(v)\,\mathcal{T}_{33}(u) - \frac{i}{u-v}\mathcal{T}_{13}(u)\,\mathcal{T}_{33}(v)\,,
\end{aligned} \tag{3.18}$$

[6] Actually, since here the eigenvalues (3.16) factor into a product

$$\Lambda(u) = \lambda(u)\dot\lambda(u)\,,$$

we encounter the following interesting subtlety. One possibility (which we believe is the correct one) is to *separately* require the cancellation of poles in $\lambda(u)$ and $\dot\lambda(u)$, which leads to (3.17). Alternatively, one could require only the cancellation of poles in $\Lambda(u)$, which is a weaker condition. This leads to additional (spurious) Bethe ansatz-like equations which couple the left and right Bethe roots. We conclude that, although the trick of obtaining the Bethe equations by requiring cancellation of poles can save a lot of effort, it should be applied with care.

[7] These equations may look strange. However, they can be recast in the more familiar (symmetric) form by shifting all the Bethe roots by $i/2$; i.e., $u_j \mapsto u_j - i/2\,, \dot u_j \mapsto \dot u_j - i/2$.

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where now the subscripts refer to matrix elements of the monodromy matrix regarded as a 4×4 matrix of operators. In each exchange relation, the first ("diagonal") term gives the "wanted" contribution, and the second term gives "unwanted" contributions. With more effort, one can also show that (see again appendix A)

$$\begin{aligned}
\mathcal{T}_{22}(u)\,\mathcal{T}_{13}(v) &= \frac{u-v-i}{u-v}\mathcal{T}_{13}(v)\,\mathcal{T}_{22}(u) + \frac{i(u-v-i)}{(u-v)^2}\mathcal{T}_{14}(v)\,\mathcal{T}_{21}(u) \\
&\quad + \frac{i}{u-v}\mathcal{T}_{24}(u)\,\mathcal{T}_{11}(v) - \frac{i}{u-v}\mathcal{T}_{12}(u)\,\mathcal{T}_{23}(v) - \frac{1}{(u-v)^2}\mathcal{T}_{14}(u)\,\mathcal{T}_{21}(v)\,, \\
\mathcal{T}_{33}(u)\,\mathcal{T}_{12}(v) &= \frac{u-v-i}{u-v}\mathcal{T}_{12}(v)\,\mathcal{T}_{33}(u) + \frac{i(u-v-i)}{(u-v)^2}\mathcal{T}_{14}(v)\,\mathcal{T}_{31}(u) \\
&\quad + \frac{i}{u-v}\mathcal{T}_{34}(u)\,\mathcal{T}_{11}(v) - \frac{i}{u-v}\mathcal{T}_{13}(u)\,\mathcal{T}_{32}(v) - \frac{1}{(u-v)^2}\mathcal{T}_{14}(u)\,\mathcal{T}_{31}(v)\,, \\
\mathcal{T}_{44}(u)\,\mathcal{T}_{12}(v) &= \frac{u-v+i}{u-v}\mathcal{T}_{12}(v)\,\mathcal{T}_{44}(u) + \frac{i(u-v+i)}{(u-v)^2}\mathcal{T}_{14}(v)\,\mathcal{T}_{42}(u) \\
&\quad - \frac{i}{u-v}\mathcal{T}_{34}(u)\,\mathcal{T}_{22}(v) - \frac{i}{u-v}\mathcal{T}_{24}(u)\,\mathcal{T}_{32}(v) + \frac{1}{(u-v)^2}\mathcal{T}_{14}(u)\,\mathcal{T}_{42}(v)\,, \\
\mathcal{T}_{44}(u)\,\mathcal{T}_{13}(v) &= \frac{u-v+i}{u-v}\mathcal{T}_{13}(v)\,\mathcal{T}_{44}(u) + \frac{i(u-v+i)}{(u-v)^2}\mathcal{T}_{14}(v)\,\mathcal{T}_{43}(u) \qquad (3.19) \\
&\quad - \frac{i}{u-v}\mathcal{T}_{24}(u)\,\mathcal{T}_{33}(v) - \frac{i}{u-v}\mathcal{T}_{34}(u)\,\mathcal{T}_{23}(v) + \frac{1}{(u-v)^2}\mathcal{T}_{14}(u)\,\mathcal{T}_{43}(v)\,,
\end{aligned}$$

where again only the first (diagonal) term gives the "wanted" contribution.

We make the ansatz that the eigenstates of the transfer matrix are given by (cf., (3.15))

$$|\Lambda\rangle = \prod_{j=1}^{m}\mathcal{T}_{13}(u_j)\prod_{k=1}^{\dot{m}}\mathcal{T}_{12}(\dot{u}_k)\left(|0\rangle\otimes|\dot{0}\rangle\right)\,, \qquad (3.20)$$

We observe that the vacuum state is an eigenstate of the diagonal elements of the monodromy matrix,

$$\begin{aligned}
\mathcal{T}_{11}(u)\left(|0\rangle\otimes|\dot{0}\rangle\right) &= S_0(u)^{2L}\,(u+i)^{2L}\left(|0\rangle\otimes|\dot{0}\rangle\right)\,, \\
\mathcal{T}_{22}(u)\left(|0\rangle\otimes|\dot{0}\rangle\right) &= S_0(u)^{2L}\,(u+i)^{L}u^{L}\left(|0\rangle\otimes|\dot{0}\rangle\right)\,, \\
\mathcal{T}_{33}(u)\left(|0\rangle\otimes|\dot{0}\rangle\right) &= S_0(u)^{2L}\,(u+i)^{L}u^{L}\left(|0\rangle\otimes|\dot{0}\rangle\right)\,, \\
\mathcal{T}_{44}(u)\left(|0\rangle\otimes|\dot{0}\rangle\right) &= S_0(u)^{2L}\,u^{2L}\left(|0\rangle\otimes|\dot{0}\rangle\right)\,, \qquad (3.21)
\end{aligned}$$

The transfer matrix is evidently given by the sum of the diagonal elements of the monodromy matrix,

$$t(u) = \mathcal{T}_{11}(u) + \mathcal{T}_{22}(u) + \mathcal{T}_{33}(u) + \mathcal{T}_{44}(u)\,. \qquad (3.22)$$

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Acting with this operator on the states (3.20), we use (in the standard way) the first term of the exchange relations (3.18), (3.19) together with (3.21) to obtain the eigenvalue,

$$
\begin{aligned}
t(u)|\Lambda\rangle = S_0(u)^{2L} \Big\{ & (u+i)^{2L} \prod_{j=1}^{m}\left(\frac{u-u_j-i}{u-u_j}\right) \prod_{k=1}^{\dot{m}}\left(\frac{u-\dot{u}_k-i}{u-\dot{u}_k}\right) \\
& +(u+i)^L u^L \prod_{j=1}^{m}\left(\frac{u-u_j-i}{u-u_j}\right) \prod_{k=1}^{\dot{m}}\left(\frac{u-\dot{u}_k+i}{u-\dot{u}_k}\right) \\
& +(u+i)^L u^L \prod_{j=1}^{m}\left(\frac{u-u_j+i}{u-u_j}\right) \prod_{k=1}^{\dot{m}}\left(\frac{u-\dot{u}_k-i}{u-\dot{u}_k}\right) \\
& +u^{2L} \prod_{j=1}^{m}\left(\frac{u-u_j+i}{u-u_j}\right) \prod_{k=1}^{\dot{m}}\left(\frac{u-\dot{u}_k+i}{u-\dot{u}_k}\right) \Big\} |\Lambda\rangle + \text{"unwanted"} \ . \qquad (3.23)
\end{aligned}
$$

We observe that the eigenvalue coincides with our previous result (3.16), and so we again obtain the Bethe equations (3.17).

3.3 Twisted case

We are finally ready to tackle the twisted case. We have verified that only the diagonal elements of the monodromy matrix $\tilde{\mathcal{T}}$ (3.9) are affected by the twist (3.6). (This result is a consequence of the special structure of the S-matrix.) Hence, we can hope that the exchange relations (3.18), (3.19) suffer only deformations of the coefficients; and this is exactly what we find. Indeed, with the help of Mathematica, we find

$$
\begin{aligned}
\tilde{\mathcal{T}}_{11}(u)\,\tilde{\mathcal{T}}_{13}(v) &= e^{i\gamma_1}\left(\frac{u-v-i}{u-v}\tilde{\mathcal{T}}_{13}(v)\,\tilde{\mathcal{T}}_{11}(u) + \frac{i}{u-v}\tilde{\mathcal{T}}_{13}(u)\,\tilde{\mathcal{T}}_{11}(v)\right) , \\
\tilde{\mathcal{T}}_{11}(u)\,\tilde{\mathcal{T}}_{12}(v) &= e^{-i\gamma_1}\left(\frac{u-v-i}{u-v}\tilde{\mathcal{T}}_{12}(v)\,\tilde{\mathcal{T}}_{11}(u) + \frac{i}{u-v}\tilde{\mathcal{T}}_{12}(u)\,\tilde{\mathcal{T}}_{11}(v)\right) , \\
\tilde{\mathcal{T}}_{22}(u)\,\tilde{\mathcal{T}}_{12}(v) &= e^{-i\gamma_1}\left(\frac{u-v+i}{u-v}\tilde{\mathcal{T}}_{12}(v)\,\tilde{\mathcal{T}}_{22}(u) - \frac{i}{u-v}\tilde{\mathcal{T}}_{12}(u)\,\tilde{\mathcal{T}}_{22}(v)\right) , \\
\tilde{\mathcal{T}}_{33}(u)\,\tilde{\mathcal{T}}_{13}(v) &= e^{i\gamma_1}\left(\frac{u-v+i}{u-v}\tilde{\mathcal{T}}_{13}(v)\,\tilde{\mathcal{T}}_{33}(u) - \frac{i}{u-v}\tilde{\mathcal{T}}_{13}(u)\,\tilde{\mathcal{T}}_{33}(v)\right) , \qquad (3.24)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\mathcal{T}}_{22}(u)\,\tilde{\mathcal{T}}_{13}(v) = e^{-i\gamma_1}\Big(&\frac{u-v-i}{u-v}\tilde{\mathcal{T}}_{13}(v)\,\tilde{\mathcal{T}}_{22}(u) + \frac{i(u-v-i)}{(u-v)^2}\tilde{\mathcal{T}}_{14}(v)\,\tilde{\mathcal{T}}_{21}(u) \\
&+e^{i\gamma_1}\frac{i}{u-v}\tilde{\mathcal{T}}_{24}(u)\,\tilde{\mathcal{T}}_{11}(v) - \frac{i}{u-v}\tilde{\mathcal{T}}_{12}(u)\,\tilde{\mathcal{T}}_{23}(v) - \frac{1}{(u-v)^2}\tilde{\mathcal{T}}_{14}(u)\,\tilde{\mathcal{T}}_{21}(v)\Big)\,, \\
\tilde{\mathcal{T}}_{33}(u)\,\tilde{\mathcal{T}}_{12}(v) = e^{i\gamma_1}\Big(&\frac{u-v-i}{u-v}\tilde{\mathcal{T}}_{12}(v)\,\tilde{\mathcal{T}}_{33}(u) + \frac{i(u-v-i)}{(u-v)^2}\tilde{\mathcal{T}}_{14}(v)\,\tilde{\mathcal{T}}_{31}(u) \\
&+e^{-i\gamma_1}\frac{i}{u-v}\tilde{\mathcal{T}}_{34}(u)\,\tilde{\mathcal{T}}_{11}(v) - \frac{i}{u-v}\tilde{\mathcal{T}}_{13}(u)\,\tilde{\mathcal{T}}_{32}(v) - \frac{1}{(u-v)^2}\tilde{\mathcal{T}}_{14}(u)\,\tilde{\mathcal{T}}_{31}(v)\Big)\,, \\
\tilde{\mathcal{T}}_{44}(u)\,\tilde{\mathcal{T}}_{12}(v) = e^{i\gamma_1}&\frac{u-v+i}{u-v}\tilde{\mathcal{T}}_{12}(v)\,\tilde{\mathcal{T}}_{44}(u) + \frac{i(u-v+i)}{(u-v)^2}\tilde{\mathcal{T}}_{14}(v)\,\tilde{\mathcal{T}}_{42}(u) \\
&-\frac{i}{u-v}\tilde{\mathcal{T}}_{34}(u)\,\tilde{\mathcal{T}}_{22}(v) - \frac{i}{u-v}\tilde{\mathcal{T}}_{24}(u)\,\tilde{\mathcal{T}}_{32}(v) + \frac{1}{(u-v)^2}\tilde{\mathcal{T}}_{14}(u)\,\tilde{\mathcal{T}}_{42}(v)\,, \\
\tilde{\mathcal{T}}_{44}(u)\,\tilde{\mathcal{T}}_{13}(v) = e^{-i\gamma_1}&\frac{u-v+i}{u-v}\tilde{\mathcal{T}}_{13}(v)\,\tilde{\mathcal{T}}_{44}(u) + \frac{i(u-v+i)}{(u-v)^2}\tilde{\mathcal{T}}_{14}(v)\,\tilde{\mathcal{T}}_{43}(u) \\
&-\frac{i}{u-v}\tilde{\mathcal{T}}_{24}(u)\,\tilde{\mathcal{T}}_{33}(v) - \frac{i}{u-v}\tilde{\mathcal{T}}_{34}(u)\,\tilde{\mathcal{T}}_{23}(v) + \frac{1}{(u-v)^2}\tilde{\mathcal{T}}_{14}(u)\,\tilde{\mathcal{T}}_{43}(v)\,. \qquad (3.25)
\end{aligned}
$$

Moreover, the vacuum eigenvalues (3.21) become

$$
\begin{aligned}
\tilde{\mathcal{T}}_{11}(u)\left(|0\rangle\otimes|\dot{0}\rangle\right) &= S_0(u)^{2L}\,(u+i)^{2L}\left(|0\rangle\otimes|\dot{0}\rangle\right)\,, \\
\tilde{\mathcal{T}}_{22}(u)\left(|0\rangle\otimes|\dot{0}\rangle\right) &= S_0(u)^{2L}\,e^{i\gamma_1 L}\,(u+i)^L u^L\left(|0\rangle\otimes|\dot{0}\rangle\right)\,, \\
\tilde{\mathcal{T}}_{33}(u)\left(|0\rangle\otimes|\dot{0}\rangle\right) &= S_0(u)^{2L}\,e^{-i\gamma_1 L}\,(u+i)^L u^L\left(|0\rangle\otimes|\dot{0}\rangle\right)\,, \\
\tilde{\mathcal{T}}_{44}(u)\left(|0\rangle\otimes|\dot{0}\rangle\right) &= S_0(u)^{2L}\,u^{2L}\left(|0\rangle\otimes|\dot{0}\rangle\right)\,. \qquad (3.26)
\end{aligned}
$$

The twisted transfer matrix (3.8) is given by

$$
\tilde{t}(u) = e^{i(\gamma_2+\gamma_3)/2}\tilde{\mathcal{T}}_{11}(u) + e^{i(\gamma_2-\gamma_3)/2}\tilde{\mathcal{T}}_{22}(u) + e^{i(\gamma_3-\gamma_2)/2}\tilde{\mathcal{T}}_{33}(u) + e^{-i(\gamma_2+\gamma_3)/2}\tilde{\mathcal{T}}_{44}(u)\,. \qquad (3.27)
$$

Using a similar ansatz as before (3.20), namely,

$$
|\tilde{\Lambda}\rangle = \prod_{j=1}^{m}\tilde{\mathcal{T}}_{13}(u_j)\prod_{k=1}^{\dot{m}}\tilde{\mathcal{T}}_{12}(\dot{u}_k)\left(|0\rangle\otimes|\dot{0}\rangle\right)\,, \qquad (3.28)
$$

we find that the eigenvalues of the twisted transfer matrix are given by

$$
\begin{aligned}
\tilde{\Lambda}(u) = S_0(u)^{2L}\Bigg\{&(u+i)^{2L}e^{i(\gamma_2+\gamma_3)/2}e^{i\gamma_1(m-\dot{m})}\prod_{j=1}^{m}\left(\frac{u-u_j-i}{u-u_j}\right)\prod_{k=1}^{\dot{m}}\left(\frac{u-\dot{u}_k-i}{u-\dot{u}_k}\right) \\
&+(u+i)^L u^L e^{i(\gamma_2-\gamma_3)/2}e^{i\gamma_1 L}\,e^{-i\gamma_1(m+\dot{m})}\prod_{j=1}^{m}\left(\frac{u-u_j-i}{u-u_j}\right)\prod_{k=1}^{\dot{m}}\left(\frac{u-\dot{u}_k+i}{u-\dot{u}_k}\right) \\
&+(u+i)^L u^L e^{i(\gamma_3-\gamma_2)/2}e^{-i\gamma_1 L}\,e^{i\gamma_1(m+\dot{m})}\prod_{j=1}^{m}\left(\frac{u-u_j+i}{u-u_j}\right)\prod_{k=1}^{\dot{m}}\left(\frac{u-\dot{u}_k-i}{u-\dot{u}_k}\right) \\
&+u^{2L}e^{-i(\gamma_2+\gamma_3)/2}e^{-i\gamma_1(m-\dot{m})}\prod_{j=1}^{m}\left(\frac{u-u_j+i}{u-u_j}\right)\prod_{k=1}^{\dot{m}}\left(\frac{u-\dot{u}_k+i}{u-\dot{u}_k}\right)\Bigg\}. \qquad (3.29)
\end{aligned}
$$

Remarkably, although the transfer matrix does not seem to factor into two pieces, the eigenvalues do:

$$\tilde{\Lambda}(u) = S_0(u)^{2L}\left[c_1(u+i)^L\prod_{j=1}^{m}\left(\frac{u-u_j-i}{u-u_j}\right)+c_1^{-1}u^L\prod_{j=1}^{m}\left(\frac{u-u_j+i}{u-u_j}\right)\right] \times\left[c_2(u+i)^L\prod_{k=1}^{\dot{m}}\left(\frac{u-\dot{u}_k-i}{u-\dot{u}_k}\right)+c_2^{-1}u^L\prod_{k=1}^{\dot{m}}\left(\frac{u-\dot{u}_k+i}{u-\dot{u}_k}\right)\right], \tag{3.30}$$

where

$$c_1 = e^{i\gamma_2/2}e^{i\gamma_1 L/2}e^{-i\gamma_1\dot{m}}\,, \quad c_2 = e^{i\gamma_3/2}e^{-i\gamma_1 L/2}e^{i\gamma_1 m}\,. \tag{3.31}$$

We can obtain the Bethe equations (as in the untwisted case) using the shortcut of requiring that the poles cancel,[8]

$$\begin{aligned}\left(\frac{u_j+i}{u_j}\right)^L &= e^{-i\gamma_2}e^{-i\gamma_1 L}\,e^{2i\gamma_1\dot{m}}\prod_{j'\neq j}^{m}\frac{u_j-u_{j'}+i}{u_j-u_{j'}-i}\,,\\ \left(\frac{\dot{u}_k+i}{\dot{u}_k}\right)^L &= e^{-i\gamma_3}e^{i\gamma_1 L}\,e^{-2i\gamma_1 m}\prod_{k'\neq k}^{\dot{m}}\frac{\dot{u}_k-\dot{u}_{k'}+i}{\dot{u}_k-\dot{u}_{k'}-i}\,.\end{aligned} \tag{3.32}$$

The eigenvalues (3.30) and Bethe equations (3.32) are the main results of this section. Gratifyingly, these results are simple deformations of the corresponding untwisted results (3.16) and (3.17), respectively.

The generalization to the case of an inhomogeneous chain, with inhomogeneity θ_l at site l, is now straightforward. It amounts to making the replacements

$$(u+i)^L \mapsto \prod_{l=1}^{L}(u-\theta_l+i)\,, \quad u^L \mapsto \prod_{l=1}^{L}(u-\theta_l)\,, \quad S_0(u)^{2L} \mapsto \prod_{l=1}^{L}S_0(u-\theta_l)^2 \tag{3.33}$$

in the expression (3.30) for the eigenvalues. Thus, the eigenvalues are given by

$$\begin{aligned}\tilde{\Lambda}(u) &= \prod_{l=1}^{L}S_0(u-\theta_l)^2 \\ &\times\left[c_1\prod_{l=1}^{L}(u-\theta_l+i)\prod_{j=1}^{m}\left(\frac{u-u_j-i}{u-u_j}\right)+c_1^{-1}\prod_{l=1}^{L}(u-\theta_l)\prod_{j=1}^{m}\left(\frac{u-u_j+i}{u-u_j}\right)\right]\\ &\times\left[c_2\prod_{l=1}^{L}(u-\theta_l+i)\prod_{k=1}^{\dot{m}}\left(\frac{u-\dot{u}_k-i}{u-\dot{u}_k}\right)+c_2^{-1}\prod_{l=1}^{L}(u-\theta_l)\prod_{k=1}^{\dot{m}}\left(\frac{u-\dot{u}_k+i}{u-\dot{u}_k}\right)\right].\end{aligned} \tag{3.34}$$

[8] The result (3.32) is similar in structure to the Bethe ansatz result of quantum braided algebras [40–43]. This may indicate that quantum braided algebras can be equivalently described by unbraided algebras, but with twisted R-matrices.

The equations for the auxiliary Bethe roots become

$$\begin{aligned}
\prod_{l=1}^{L}\left(\frac{u_j-\theta_l+i}{u_j-\theta_l}\right) &= e^{-i\gamma_2}e^{-i\gamma_1 L}\,e^{2i\gamma_1\dot{m}}\prod_{j'\neq j}^{m}\frac{u_j-u_{j'}+i}{u_j-u_{j'}-i}\,,\\
\prod_{l=1}^{L}\left(\frac{\dot{u}_k-\theta_l+i}{\dot{u}_k-\theta_l}\right) &= e^{-i\gamma_3}e^{i\gamma_1 L}\,e^{-2i\gamma_1 m}\prod_{k'\neq k}^{\dot{m}}\frac{\dot{u}_k-\dot{u}_{k'}+i}{\dot{u}_k-\dot{u}_{k'}-i}\,.
\end{aligned} \tag{3.35}$$

Finally, the Bethe-Yang equations corresponding to the middle node are given by

$$\begin{aligned}
e^{-ip_k\mathcal{L}} &= \tilde{\Lambda}(\theta_k)\\
&= -e^{i(\gamma_2+\gamma_3)/2}e^{i\gamma_1(m-\dot{m})}\prod_{l\neq k}^{L}S_0(\theta_k-\theta_l)^2(\theta_k-\theta_l+i)^2\\
&\quad\times\prod_{j=1}^{m}\left(\frac{\theta_k-u_j-i}{\theta_k-u_j}\right)\prod_{k=1}^{\dot{m}}\left(\frac{\theta_k-\dot{u}_k-i}{\theta_k-\dot{u}_k}\right)\,,
\end{aligned} \tag{3.36}$$

where $\mathcal{L}$ is the length of the ring with the L particles of rapidities $\theta_1,\ldots,\theta_L$.

4 Twisting two copies of the Hubbard model

We have seen that, for the $su(2)$ principal chiral model, we were able to obtain the Bethe equations for the case of a non-factoring twist by developing an algebraic Bethe ansatz based on the "full" monodromy matrix of the two S-matrix factors. In this section we shall follow the same approach for two copies of the Hubbard model with a non-factoring twist, which is technically very similar to the twisted AdS_5/CFT_4 problem. For a single copy of the Hubbard model, the algebraic Bethe ansatz was worked out in the paper by Martins and Ramos [32], which hereafter we denote by MR.

Let $S(\lambda,\mu)$ be the 16×16 R-matrix of the Hubbard model, which was found by Shastry [48, 49]. In the notation of MR

$$S(\lambda,\mu)=\mathcal{P}R_g(\lambda,\mu)\,, \tag{4.1}$$

where $R_g(\lambda,\mu)$ is given in MR (18), and $\mathcal{P}$ is the graded permutation matrix,

$$\mathcal{P}=\sum_{a,b=1}^{4}(-1)^{p(a)p(b)}e_{ab}\otimes e_{ba}\,, \tag{4.2}$$

where the gradings are given by

$$p(1)=p(4)=0\,,\qquad p(2)=p(3)=1\,, \tag{4.3}$$

and e_{ab} are the standard elementary matrices with matrix elements

$$(e_{ab})_{ij}=\delta_{a,i}\delta_{b,j}\,. \tag{4.4}$$

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Paralleling our discussion of the $su(2)$ principal chiral model, we introduce the tensor product of two such S-matrices,

$$\mathcal{S}_{a\,\dot{a}\,b\,\dot{b}}(\lambda,\mu) = S_{ab}(\lambda,\mu)\, S_{\dot{a}\dot{b}}(\lambda,\mu)\,. \tag{4.5}$$

We consider the Drinfeld-Reshetikhin-twisted S-matrix

$$\tilde{\mathcal{S}}(\lambda,\mu) = F\,\mathcal{S}(\lambda,\mu)\,F\,, \tag{4.6}$$

where the twist matrix F is again given by

$$F = e^{i\gamma_1(h\otimes\mathbb{I}\otimes\mathbb{I}\otimes h - \mathbb{I}\otimes h\otimes h\otimes\mathbb{I})}\,, \tag{4.7}$$

except h is now the diagonal matrix

$$h = \text{diag}\left(\frac{1}{2},0,0,-\frac{1}{2}\right), \tag{4.8}$$

and $\mathbb{I}$ is the 4×4 unit matrix. Note that F cannot be factored into matrices acting separately on the 13 space and the 24 space.

The main problem is to determine the eigenvalues of the transfer matrix

$$\tilde{t}(\lambda) = \text{str}_{a\dot{a}}\, M_{a\dot{a}} \tilde{\mathcal{T}}_{a\dot{a}}(\lambda)\,, \tag{4.9}$$

where the monodromy matrix is given by

$$\tilde{\mathcal{T}}_{a\dot{a}}(\lambda) = \tilde{\mathcal{S}}_{a\,\dot{a}\,1\,\dot{1}}(\lambda,0)\cdots\tilde{\mathcal{S}}_{a\,\dot{a}\,L\,\dot{L}}(\lambda,0)\,, \tag{4.10}$$

and the matrix $M_{a\dot{a}}$ is given by

$$M = e^{i\gamma_2 h}\otimes e^{i\gamma_3 h}\,. \tag{4.11}$$

We follow the same approach as for the twisted $su(2)$ principal chiral model. The first step is to understand the untwisted case.

4.1 Untwisted case

We now consider the untwisted case; i.e., $\gamma_i = 0$, and therefore both F and M are 1. In this case, the monodromy matrix is given by (see (3.12)),

$$\mathcal{T}_{a\dot{a}}(\lambda) = T_a(\lambda)\; T_{\dot{a}}(\lambda)\,. \tag{4.12}$$

We label the matrix elements of $T_a(\lambda)$ as in MR (21); and similarly for the matrix elements of $T_{\dot{a}}(\lambda)$, except we also decorate those with a dot. Hence,

$$\mathcal{T}(\lambda) = \begin{pmatrix} B & B_1 & B_2 & F \\ C_1 & A_{11} & A_{12} & B_1^* \\ C_2 & A_{21} & A_{22} & B_2^* \\ C & C_1^* & C_2^* & D \end{pmatrix} \otimes \begin{pmatrix} \dot{B} & \dot{B}_1 & \dot{B}_2 & \dot{F} \\ \dot{C}_1 & \dot{A}_{11} & \dot{A}_{12} & \dot{B}_1^* \\ \dot{C}_2 & \dot{A}_{21} & \dot{A}_{22} & \dot{B}_2^* \\ \dot{C} & \dot{C}_1^* & \dot{C}_2^* & \dot{D} \end{pmatrix}\,. \tag{4.13}$$

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We regard $\mathcal{T}(\lambda)$ as a 16×16 matrix of operators. We denote these operators by their matrix elements, i.e., $\mathcal{T}_{j,k}(\lambda)$, where $j, k \in \{1, \ldots, 16\}$. In particular,

$$\begin{aligned} \mathcal{T}_{1,1}(\lambda) &= B(\lambda)\, \dot{B}(\lambda)\,, \qquad \mathcal{T}_{1,2}(\lambda) = B(\lambda)\, \dot{B}_1(\lambda)\,, \qquad \mathcal{T}_{1,3}(\lambda) = B(\lambda)\, \dot{B}_2(\lambda)\,, \\ \mathcal{T}_{1,5}(\lambda) &= B_1(\lambda)\, \dot{B}(\lambda)\,, \qquad \mathcal{T}_{1,9}(\lambda) = B_2(\lambda)\, \dot{B}(\lambda)\,. \end{aligned} \tag{4.14}$$

Since the one-particle states are given by MR (44), it is reasonable to consider the tensor-product states

$$|\Phi\rangle = \left[\mathcal{T}_{1,5}(\lambda_1)\mathcal{F}^1 + \mathcal{T}_{1,9}(\lambda_1)\mathcal{F}^2\right] \left[\mathcal{T}_{1,2}(\dot{\lambda}_1)\dot{\mathcal{F}}^1 + \mathcal{T}_{1,3}(\dot{\lambda}_1)\dot{\mathcal{F}}^2\right] \left(|0\rangle \otimes |\dot{0}\rangle\right) . \tag{4.15}$$

The exchange relations of the diagonal elements $\mathcal{T}_{j,j}(\lambda)$ with the creation operators $\mathcal{T}_{1,2}(\mu), \mathcal{T}_{1,3}(\mu), \mathcal{T}_{1,5}(\mu), \mathcal{T}_{1,9}(\mu)$ are discussed in appendix A. We explicitly record here just the "wanted" (diagonal) terms,

$$\begin{aligned} \mathcal{T}_{j,j}(\lambda)\, \mathcal{T}_{1,2}(\mu) &= f_{l(j)}(\lambda,\mu)\, \mathcal{T}_{1,2}(\mu)\, \mathcal{T}_{j,j}(\lambda) + \ldots\,, \\ \mathcal{T}_{j,j}(\lambda)\, \mathcal{T}_{1,3}(\mu) &= g_{l(j)}(\lambda,\mu)\, \mathcal{T}_{1,3}(\mu)\, \mathcal{T}_{j,j}(\lambda) + \ldots\,, \\ \mathcal{T}_{j,j}(\lambda)\, \mathcal{T}_{1,5}(\mu) &= f_{k(j)}(\lambda,\mu)\, \mathcal{T}_{1,5}(\mu)\, \mathcal{T}_{j,j}(\lambda) + \ldots\,, \\ \mathcal{T}_{j,j}(\lambda)\, \mathcal{T}_{1,9}(\mu) &= g_{k(j)}(\lambda,\mu)\, \mathcal{T}_{1,9}(\mu)\, \mathcal{T}_{j,j}(\lambda) + \ldots\,, \quad j = 1, \ldots, 16, \end{aligned} \tag{4.16}$$

where $k(j)$ and $l(j)$ are functions of j such that[9]

$$j = 4(k-1) + l\,, \qquad j = 1, \ldots, 16, \qquad k\,, l \in \{1,2,3,4\}\,, \tag{4.17}$$

and we have defined $f_i(\lambda,\mu)$ and $g_i(\lambda,\mu)$ by

$$\begin{aligned} f_1(\lambda,\mu) &= g_1(\lambda,\mu) = \frac{i\alpha_2(\mu,\lambda)}{\alpha_9(\mu,\lambda)}\,, \\ f_2(\lambda,\mu) &= g_3(\lambda,\mu) = -\frac{i\alpha_1(\lambda,\mu)}{\alpha_9(\lambda,\mu)}\,, \\ f_3(\lambda,\mu) &= g_2(\lambda,\mu) = -\frac{i\alpha_1(\lambda,\mu)}{\alpha_9(\lambda,\mu)}\bar{b}(\lambda,\mu)\,, \\ f_4(\lambda,\mu) &= g_4(\lambda,\mu) = -\frac{i\alpha_8(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\,. \end{aligned} \tag{4.18}$$

Moreover, $\alpha_i(\lambda,\mu)$ are defined in MR (A.1)-(A.9), and $\bar{b}(\lambda,\mu)$ is defined in MR (27).

We observe that the pseudovacuum state $|0\rangle \otimes |\dot{0}\rangle$ is an eigenstate of the diagonal elements of the monodromy matrix,

$$\mathcal{T}_{j,j}(\lambda) \left(|0\rangle \otimes |\dot{0}\rangle\right) = \left[\phi_{k(j)}(\lambda)\, \phi_{l(j)}(\lambda)\right]^L \left(|0\rangle \otimes |\dot{0}\rangle\right)\,, \quad j = 1, \ldots, 16, \tag{4.19}$$

where we have defined $\phi_i(\lambda)$ by

$$\phi_1(\lambda) = \omega_1(\lambda)\,, \quad \phi_2(\lambda) = \phi_3(\lambda) = \omega_2(\lambda)\,, \quad \phi_4(\lambda) = \omega_3(\lambda)\,, \tag{4.20}$$

[9]Hence, $(k,l) = (1,1)$ for $j=1$; and $(k,l) = (1,2)$ for $j=2$, etc.

and $\omega_i(\lambda)$ are given by

$$\omega_1(\lambda) = \alpha_2(\lambda, 0)\,, \quad \omega_2(\lambda) = -i\alpha_9(\lambda, 0)\,, \quad \omega_3(\lambda) = \alpha_7(\lambda, 0)\,. \tag{4.21}$$

The transfer matrix is given by

$$\begin{aligned} t(\lambda) = \ & \mathcal{T}_{1,1}(\lambda) - \mathcal{T}_{2,2}(\lambda) - \mathcal{T}_{3,3}(\lambda) + \mathcal{T}_{4,4}(\lambda) \\ & - [\mathcal{T}_{5,5}(\lambda) - \mathcal{T}_{6,6}(\lambda) - \mathcal{T}_{7,7}(\lambda) + \mathcal{T}_{8,8}(\lambda)] \\ & - [\mathcal{T}_{9,9}(\lambda) - \mathcal{T}_{10,10}(\lambda) - \mathcal{T}_{11,11}(\lambda) + \mathcal{T}_{12,12}(\lambda)] \\ & + \mathcal{T}_{13,13}(\lambda) - \mathcal{T}_{14,14}(\lambda) - \mathcal{T}_{15,15}(\lambda) + \mathcal{T}_{16,16}(\lambda)\,, \end{aligned} \tag{4.22}$$

where the signs are due to the supertrace. Acting with this operator on the states (4.15), we use (in the standard way) the first term of the exchange relations (4.16), together with the pseudovacuum eigenvalues (4.19), to obtain the eigenvalue. Upon factoring the result, we obtain

$$\begin{aligned} \Lambda(\lambda) = & \left[\omega_1(\lambda)^L \left(\frac{i\alpha_2(\lambda_1, \lambda)}{\alpha_9(\lambda_1, \lambda)}\right) + \omega_3(\lambda)^L \left(\frac{-i\alpha_8(\lambda, \lambda_1)}{\alpha_7(\lambda, \lambda_1)}\right) \right. \\ & \left. -\omega_2(\lambda)^L \left(\frac{-i\alpha_1(\lambda, \lambda_1)}{\alpha_9(\lambda, \lambda_1)}\right) \Lambda^{(1)}(\lambda, \lambda_1)\right] \\ & \times \left[\omega_1(\lambda)^L \left(\frac{i\alpha_2(\dot{\lambda}_1, \lambda)}{\alpha_9(\dot{\lambda}_1, \lambda)}\right) + \omega_3(\lambda)^L \left(\frac{-i\alpha_8(\lambda, \dot{\lambda}_1)}{\alpha_7(\lambda, \dot{\lambda}_1)}\right) \right. \\ & \left. -\omega_2(\lambda)^L \left(\frac{-i\alpha_1(\lambda, \dot{\lambda}_1)}{\alpha_9(\lambda, \dot{\lambda}_1)}\right) \Lambda^{(1)}(\lambda, \dot{\lambda}_1)\right], \end{aligned} \tag{4.23}$$

where, as in MR (51),

$$\Lambda^{(1)}(\lambda, \lambda_1) = 1 + \bar{b}(\lambda, \lambda_1)\,, \qquad \Lambda^{(1)}(\lambda, \dot{\lambda}_1) = 1 + \bar{b}(\lambda, \dot{\lambda}_1)\,. \tag{4.24}$$

On the basis of MR, we assume that this result can be extrapolated to the general case:

$$\begin{aligned} \Lambda(\lambda) = & \left[\omega_1(\lambda)^L \prod_{j=1}^{n} \left(\frac{i\alpha_2(\lambda_j, \lambda)}{\alpha_9(\lambda_j, \lambda)}\right) + \omega_3(\lambda)^L \prod_{j=1}^{n} \left(\frac{-i\alpha_8(\lambda, \lambda_j)}{\alpha_7(\lambda, \lambda_j)}\right) \right. \\ & \left. -\omega_2(\lambda)^L \prod_{j=1}^{n} \left(\frac{-i\alpha_1(\lambda, \lambda_j)}{\alpha_9(\lambda, \lambda_j)}\right) \Lambda^{(1)}(\lambda, \{\lambda_k\}) \right] \\ & \times \left[\omega_1(\lambda)^L \prod_{j=1}^{\dot{n}} \left(\frac{i\alpha_2(\dot{\lambda}_j, \lambda)}{\alpha_9(\dot{\lambda}_j, \lambda)}\right) + \omega_3(\lambda)^L \prod_{j=1}^{\dot{n}} \left(\frac{-i\alpha_8(\lambda, \dot{\lambda}_j)}{\alpha_7(\lambda, \dot{\lambda}_j)}\right) \right. \\ & \left. -\omega_2(\lambda)^L \prod_{j=1}^{\dot{n}} \left(\frac{-i\alpha_1(\lambda, \dot{\lambda}_j)}{\alpha_9(\lambda, \dot{\lambda}_j)}\right) \Lambda^{(1)}(\lambda, \{\dot{\lambda}_k\})\right]. \end{aligned} \tag{4.25}$$

As expected, this is just the product of two copies of the result for a single copy of Hubbard, given in MR (89). Note that $\Lambda^{(1)}(\lambda, \{\lambda_k\})$ are the eigenvalues of the auxiliary transfer matrix MR (92). They are given by MR (99), and there is a similar expression for $\Lambda^{(1)}(\lambda, \{\dot{\lambda}_k\})$.

The Bethe equations can again be obtained using the shortcut that there should be a cancellation of the poles in the eigenvalues of the transfer matrix; this leads to MR (90), (102),

$$
\begin{aligned}
\left(\frac{\omega_1(\lambda_j)}{\omega_2(\lambda_j)}\right)^L &= \Lambda^{(1)}(\lambda_j, \{\lambda_k\}) = \prod_{l=1}^{m} \frac{1}{\bar{b}(\mu_l, \lambda_j)}\,, \\
\left(\frac{\omega_1(\dot\lambda_j)}{\omega_2(\dot\lambda_j)}\right)^L &= \Lambda^{(1)}(\dot\lambda_j, \{\dot\lambda_k\}) = \prod_{l=1}^{\dot m} \frac{1}{\bar{b}(\dot\mu_l, \dot\lambda_j)}\,.
\end{aligned} \tag{4.26}
$$

To obtain this result, one needs the following properties of the functions α_i

$$
\alpha_9(\mu, \lambda) = -\alpha_9(\lambda, \mu)\,, \quad \alpha_1(\mu, \lambda) = \alpha_2(\lambda, \mu)\,, \tag{4.27}
$$

which follow from their definitions, given in MR appendix A. The Bethe equations for the auxiliary problem are given by MR (100), and a similar set for the dotted roots.

4.2 Twisted case

We turn now to the twisted case, for which the monodromy matrix is given by (4.10). The exchange relations suffer only deformations of the coefficients. In particular, the "wanted" terms become (cf. (4.16))

$$
\begin{aligned}
\tilde{\mathcal{T}}_{j,j}(\lambda)\, \tilde{\mathcal{T}}_{1,2}(\mu) &= e^{-i\zeta_{k(j)}\gamma_1/2} f_{l(j)}(\lambda, \mu)\, \tilde{\mathcal{T}}_{1,2}(\mu)\, \tilde{\mathcal{T}}_{j,j}(\lambda) + \ldots\,, \\
\tilde{\mathcal{T}}_{j,j}(\lambda)\, \tilde{\mathcal{T}}_{1,3}(\mu) &= e^{-i\zeta_{k(j)}\gamma_1/2} g_{l(j)}(\lambda, \mu)\, \tilde{\mathcal{T}}_{1,3}(\mu)\, \tilde{\mathcal{T}}_{j,j}(\lambda) + \ldots\,, \\
\tilde{\mathcal{T}}_{j,j}(\lambda)\, \tilde{\mathcal{T}}_{1,5}(\mu) &= e^{i\zeta_{l(j)}\gamma_1/2} f_{k(j)}(\lambda, \mu)\, \tilde{\mathcal{T}}_{1,5}(\mu)\, \tilde{\mathcal{T}}_{j,j}(\lambda) + \ldots\,, \\
\tilde{\mathcal{T}}_{j,j}(\lambda)\, \tilde{\mathcal{T}}_{1,9}(\mu) &= e^{i\zeta_{l(j)}\gamma_1/2} g_{k(j)}(\lambda, \mu)\, \tilde{\mathcal{T}}_{1,9}(\mu)\, \tilde{\mathcal{T}}_{j,j}(\lambda) + \ldots\,, \quad j = 1, \ldots, 16,
\end{aligned} \tag{4.28}
$$

where we have defined

$$
\zeta_1 = 1\,, \quad \zeta_2 = \zeta_3 = 0\,, \quad \zeta_4 = -1\,. \tag{4.29}
$$

The pseudovacuum eigenvalues are now given by (cf. (4.19))

$$
\tilde{\mathcal{T}}_{j,j}(\lambda)\, \left(|0\rangle \otimes |\dot 0\rangle\right) = e^{i(\zeta_{k(j)} - \zeta_{l(j)})\gamma_1 L/2} \left[\phi_{k(j)}(\lambda)\, \phi_{l(j)}(\lambda)\right]^L \left(|0\rangle \otimes |\dot 0\rangle\right)\,, \quad j = 1, \ldots, 16. \tag{4.30}
$$

We remind the reader that $k(j)$ and $l(j)$ are defined by (4.17).

Proceeding as in the untwisted case, one easily finds that the eigenvalues of the twisted transfer matrix (4.9) are given by

$$
\begin{aligned}
\tilde\Lambda(\lambda) = &\left[c_1\, \omega_1(\lambda)^L \prod_{j=1}^{n} \left(\frac{i\alpha_2(\lambda_j, \lambda)}{\alpha_9(\lambda_j, \lambda)}\right) + c_1^{-1}\, \omega_3(\lambda)^L \prod_{j=1}^{n} \left(\frac{-i\alpha_8(\lambda, \lambda_j)}{\alpha_7(\lambda, \lambda_j)}\right)\right. \\
&\left. -\omega_2(\lambda)^L \prod_{j=1}^{n} \left(\frac{-i\alpha_1(\lambda, \lambda_j)}{\alpha_9(\lambda, \lambda_j)}\right) \Lambda^{(1)}(\lambda, \{\lambda_k\})\right] \\
\times &\left[c_2\, \omega_1(\lambda)^L \prod_{j=1}^{\dot n} \left(\frac{i\alpha_2(\dot\lambda_j, \lambda)}{\alpha_9(\dot\lambda_j, \lambda)}\right) + c_2^{-1}\, \omega_3(\lambda)^L \prod_{j=1}^{\dot n} \left(\frac{-i\alpha_8(\lambda, \dot\lambda_j)}{\alpha_7(\lambda, \dot\lambda_j)}\right)\right. \\
&\left. -\omega_2(\lambda)^L \prod_{j=1}^{\dot n} \left(\frac{-i\alpha_1(\lambda, \dot\lambda_j)}{\alpha_9(\lambda, \dot\lambda_j)}\right) \Lambda^{(1)}(\lambda, \{\dot\lambda_k\})\right]\,,
\end{aligned} \tag{4.31}
$$

where

$$c_1 = e^{i\gamma_2/2} e^{i\gamma_1 L/2} e^{-i\gamma_1 \dot{n}/2}\,, \quad c_2 = e^{i\gamma_3/2} e^{-i\gamma_1 L/2} e^{i\gamma_1 n/2}\,. \tag{4.32}$$

It is very important to observe that the matrix $\hat{r}$ given in MR (26) does *not* get deformed by the twist. Hence, the eigenvalues of the auxiliary transfer matrix also do not get deformed. Again using the shortcut, we find from (4.31) that the Bethe equations are given by

$$\begin{aligned} e^{i\gamma_2/2} e^{i\gamma_1 L/2} e^{-i\gamma_1 \dot{n}/2} \left(\frac{\omega_1(\lambda_j)}{\omega_2(\lambda_j)}\right)^L &= \Lambda^{(1)}(\lambda_j, \{\lambda_k\}) = \prod_{l=1}^{m} \frac{1}{\bar{b}(\mu_l, \lambda_j)}\,, \\ e^{i\gamma_3/2} e^{-i\gamma_1 L/2} e^{i\gamma_1 n/2} \left(\frac{\omega_1(\dot{\lambda}_j)}{\omega_2(\dot{\lambda}_j)}\right)^L &= \Lambda^{(1)}(\dot{\lambda}_j, \{\dot{\lambda}_k\}) = \prod_{l=1}^{\dot{m}} \frac{1}{\bar{b}(\dot{\mu}_l, \dot{\lambda}_j)}\,. \end{aligned} \tag{4.33}$$

Their structure is similar to those for the twisted $su(2)$ principal chiral model (3.32). The Bethe equations for the auxiliary problem are again given by MR (100).

5 Twisting AdS_5/CFT_4

We finally come to the AdS/CFT case. Let $S(p_1, p_2)$ be the graded $su(2|2)$ S-matrix in [22, 27], and let $\mathcal{S}(p_1, p_2)$ be the $su(2|2)^2$ S-matrix,

$$\mathcal{S}_{a\,\dot{a}\,b\,\dot{b}}(p_1, p_2) = S_{ab}(p_1, p_2)\, S_{\dot{a}\dot{b}}(p_1, p_2)\,. \tag{5.1}$$

We consider the Drinfeld-Reshetikhin twist of this S-matrix

$$\tilde{\mathcal{S}}(p_1, p_2) = F\, \mathcal{S}(p_1, p_2)\, F\,, \tag{5.2}$$

where the twist matrix F is given by (see (2.23))

$$F = e^{i\gamma_1 (h\otimes\mathbb{I}\otimes\mathbb{I}\otimes h - \mathbb{I}\otimes h\otimes h\otimes\mathbb{I})}\,. \tag{5.3}$$

Here h is the diagonal matrix[10]

$$h = \text{diag}\left(\frac{1}{2}, -\frac{1}{2}, 0, 0\right), \tag{5.4}$$

and $\mathbb{I}$ is again the 4×4 unit matrix. In appendix B we verify that the twisted S-matrix has the standard crossing symmetry; hence, the scalar factor S_0 is exactly the same as the untwisted one and independent of the deformation parameter. The (inhomogeneous) transfer matrix is given by

$$\tilde{t}(\lambda) = \text{str}_{a\dot{a}}\, M_{a\dot{a}} \tilde{S}_{a\dot{a}1\dot{1}}(\lambda, p_1) \ldots \tilde{S}_{a\dot{a}N\dot{N}}(\lambda, p_N)\,, \tag{5.5}$$

[10] The difference between (4.8) and (5.4) is due to a difference in basis: as already noted in (4.3), in [32] the gradings are (0,1,1,0), while in [22] the gradings are (0,0,1,1).

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where the matrix $M_{a\dot{a}}$ is given by (see (2.24))

$$M = e^{i(\gamma_3-\gamma_2)Jh} \otimes e^{i(\gamma_3+\gamma_2)Jh}\,, \tag{5.6}$$

and J is the angular momentum charge.

Let us briefly recall the untwisted case $\gamma_i = 0$. Based on the Hubbard-model results in MR [32], Martins and Melo obtain in [27] the eigenvalues of the $su(2|2)$ transfer matrix. Hereafter we denote the latter reference by MM. Indeed, the eigenvalues of the transfer matrix for a single copy of Hubbard are given by MR (89), MR (99), which can be rewritten as MM (30). Martins and Melo argue that MM (30) leads to the eigenvalues of the $su(2|2)$ transfer matrix in MM (32).

Turning now to the twisted case, we use the same logic to conclude that our result (4.31) for twisted Hubbard implies that the eigenvalues of the twisted AdS/CFT transfer matrix (5.5) are given by (cf. MM (32))

$$\begin{aligned}
\tilde{\Lambda}(\lambda) = \prod_{i=1}^{N} S_0(\lambda,p_i)^2 \Bigg[& c_1 \prod_{i=1}^{N}\left[\frac{x^-(p_i)-x^+(\lambda)}{x^+(p_i)-x^-(\lambda)}\right]\frac{\eta(p_i)}{\eta(\lambda)} \prod_{j=1}^{m_1}\eta(\lambda)\frac{x^-(\lambda)-x^+(\lambda_j)}{x^+(\lambda)-x^+(\lambda_j)} \\
& -\prod_{i=1}^{N}\frac{x^+(\lambda)-x^+(p_i)}{x^-(\lambda)-x^+(p_i)}\frac{1}{\eta(\lambda)}\left\{\prod_{j=1}^{m_1}\eta(\lambda)\left[\frac{x^-(\lambda)-x^+(\lambda_j)}{x^+(\lambda)-x^+(\lambda_j)}\right]\prod_{l=1}^{m_2}\frac{x^+(\lambda)+\frac{1}{x^+(\lambda)}-\tilde{\mu}_l+\frac{i}{2g}}{x^+(\lambda)+\frac{1}{x^+(\lambda)}-\tilde{\mu}_l-\frac{i}{2g}}\right. \\
& \left. +\prod_{j=1}^{m_1}\eta(\lambda)\left[\frac{x^+(\lambda_j)-\frac{1}{x^+(\lambda)}}{x^+(\lambda_j)-\frac{1}{x^-(\lambda)}}\right]\prod_{l=1}^{m_2}\frac{x^-(\lambda)+\frac{1}{x^-(\lambda)}-\tilde{\mu}_l-\frac{i}{2g}}{x^-(\lambda)+\frac{1}{x^-(\lambda)}-\tilde{\mu}_l+\frac{i}{2g}}\right\} \\
& +c_1^{-1}\prod_{i=1}^{N}\left[\frac{1-\frac{1}{x^-(\lambda)x^+(p_i)}}{1-\frac{1}{x^-(\lambda)x^-(p_i)}}\right]\left[\frac{x^+(p_i)-x^+(\lambda)}{x^+(p_i)-x^-(\lambda)}\right]\frac{\eta(p_i)}{\eta(\lambda)}\prod_{j=1}^{m_1}\eta(\lambda)\left[\frac{x^+(\lambda_j)-\frac{1}{x^+(\lambda)}}{x^+(\lambda_j)-\frac{1}{x^-(\lambda)}}\right]\Bigg] \\
\times\Bigg[& c_2 \prod_{i=1}^{N}\left[\frac{x^-(p_i)-x^+(\lambda)}{x^+(p_i)-x^-(\lambda)}\right]\frac{\eta(p_i)}{\eta(\lambda)} \prod_{j=1}^{\dot{m}_1}\eta(\lambda)\frac{x^-(\lambda)-x^+(\dot{\lambda}_j)}{x^+(\lambda)-x^+(\dot{\lambda}_j)} \\
& -\prod_{i=1}^{N}\frac{x^+(\lambda)-x^+(p_i)}{x^-(\lambda)-x^+(p_i)}\frac{1}{\eta(\lambda)}\left\{\prod_{j=1}^{\dot{m}_1}\eta(\lambda)\left[\frac{x^-(\lambda)-x^+(\dot{\lambda}_j)}{x^+(\lambda)-x^+(\dot{\lambda}_j)}\right]\prod_{l=1}^{\dot{m}_2}\frac{x^+(\lambda)+\frac{1}{x^+(\lambda)}-\dot{\tilde{\mu}}_l+\frac{i}{2g}}{x^+(\lambda)+\frac{1}{x^+(\lambda)}-\dot{\tilde{\mu}}_l-\frac{i}{2g}}\right. \\
& \left. +\prod_{j=1}^{\dot{m}_1}\eta(\lambda)\left[\frac{x^+(\dot{\lambda}_j)-\frac{1}{x^+(\lambda)}}{x^+(\dot{\lambda}_j)-\frac{1}{x^-(\lambda)}}\right]\prod_{l=1}^{\dot{m}_2}\frac{x^-(\lambda)+\frac{1}{x^-(\lambda)}-\dot{\tilde{\mu}}_l-\frac{i}{2g}}{x^-(\lambda)+\frac{1}{x^-(\lambda)}-\dot{\tilde{\mu}}_l+\frac{i}{2g}}\right\} \\
& +c_2^{-1}\prod_{i=1}^{N}\left[\frac{1-\frac{1}{x^-(\lambda)x^+(p_i)}}{1-\frac{1}{x^-(\lambda)x^-(p_i)}}\right]\left[\frac{x^+(p_i)-x^+(\lambda)}{x^+(p_i)-x^-(\lambda)}\right]\frac{\eta(p_i)}{\eta(\lambda)}\prod_{j=1}^{\dot{m}_1}\eta(\lambda)\left[\frac{x^+(\dot{\lambda}_j)-\frac{1}{x^+(\lambda)}}{x^+(\dot{\lambda}_j)-\frac{1}{x^-(\lambda)}}\right]\Bigg]\,,
\end{aligned} \tag{5.7}$$

where

$$c_1 = e^{i\gamma_1 N/2}e^{-i\gamma_1\dot{m}_1/2}e^{i(\gamma_3-\gamma_2)J/2}\,, \quad c_2 = e^{-i\gamma_1 N/2}e^{i\gamma_1 m_1/2}e^{i(\gamma_3+\gamma_2)J/2}\,. \tag{5.8}$$

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The corresponding Bethe equations are therefore (cf. MM (33)):

$$
\begin{aligned}
c_1 \prod_{i=1}^{N} \left[\frac{x^+(\lambda_j) - x^-(p_i)}{x^+(\lambda_j) - x^+(p_i)} \right] \eta(p_i) &= \prod_{l=1}^{m_2} \frac{x^+(\lambda_j) + \frac{1}{x^+(\lambda_j)} - \tilde{\mu}_l + \frac{i}{2g}}{x^+(\lambda_j) + \frac{1}{x^+(\lambda_j)} - \tilde{\mu}_l - \frac{i}{2g}} \,, \\
& \qquad j = 1, \ldots, m_1 , \\
c_2 \prod_{i=1}^{N} \left[\frac{x^+(\dot{\lambda}_j) - x^-(p_i)}{x^+(\dot{\lambda}_j) - x^+(p_i)} \right] \eta(p_i) &= \prod_{l=1}^{\dot{m}_2} \frac{x^+(\dot{\lambda}_j) + \frac{1}{x^+(\dot{\lambda}_j)} - \dot{\tilde{\mu}}_l + \frac{i}{2g}}{x^+(\dot{\lambda}_j) + \frac{1}{x^+(\dot{\lambda}_j)} - \dot{\tilde{\mu}}_l - \frac{i}{2g}} \,, \\
& \qquad j = 1, \ldots, \dot{m}_1 , \\
\prod_{j=1}^{m_1} \frac{\tilde{\mu}_l - x^+(\lambda_j) - \frac{1}{x^+(\lambda_j)} + \frac{i}{2g}}{\tilde{\mu}_l - x^+(\lambda_j) - \frac{1}{x^+(\lambda_j)} - \frac{i}{2g}} &= \prod_{\substack{k=1\\k\neq l}}^{m_2} \frac{\tilde{\mu}_l - \tilde{\mu}_k + \frac{i}{g}}{\tilde{\mu}_l - \tilde{\mu}_k - \frac{i}{g}} \,, \quad l = 1, \ldots, m_2 , \\
\prod_{j=1}^{\dot{m}_1} \frac{\dot{\tilde{\mu}}_l - x^+(\dot{\lambda}_j) - \frac{1}{x^+(\dot{\lambda}_j)} + \frac{i}{2g}}{\dot{\tilde{\mu}}_l - x^+(\dot{\lambda}_j) - \frac{1}{x^+(\dot{\lambda}_j)} - \frac{i}{2g}} &= \prod_{\substack{k=1\\k\neq l}}^{\dot{m}_2} \frac{\dot{\tilde{\mu}}_l - \dot{\tilde{\mu}}_k + \frac{i}{g}}{\dot{\tilde{\mu}}_l - \dot{\tilde{\mu}}_k - \frac{i}{g}} \,, \quad l = 1, \ldots, \dot{m}_2 .
\end{aligned}
\tag{5.9}
$$

In terms of the notation in MM (45), (46), and using the fact $\eta(\lambda) = e^{i\lambda/2}$, these Bethe equations become[11]

$$
c_1 e^{i\frac{P}{2}} \prod_{i=1}^{N} \left[\frac{x^+(\lambda_j^{(1)}) - x^-(p_i)}{x^+(\lambda_j^{(1)}) - x^+(p_i)} \right] = \prod_{l=1}^{m_2^{(1)}} \frac{x^+(\lambda_j^{(1)}) + \frac{1}{x^+(\lambda_j^{(1)})} - \tilde{\mu}_l{}^{(1)} + \frac{i}{2g}}{x^+(\lambda_j^{(1)}) + \frac{1}{x^+(\lambda_j^{(1)})} - \tilde{\mu}_l{}^{(1)} - \frac{i}{2g}} \,, \qquad j = 1, \ldots, m_1^{(1)} , \tag{5.10}
$$

$$
c_2 e^{i\frac{P}{2}} \prod_{i=1}^{N} \left[\frac{x^+(\lambda_j^{(2)}) - x^-(p_i)}{x^+(\lambda_j^{(2)}) - x^+(p_i)} \right] = \prod_{l=1}^{m_2^{(2)}} \frac{x^+(\lambda_j^{(2)}) + \frac{1}{x^+(\lambda_j^{(2)})} - \tilde{\mu}_l{}^{(2)} + \frac{i}{2g}}{x^+(\lambda_j^{(2)}) + \frac{1}{x^+(\lambda_j^{(2)})} - \tilde{\mu}_l{}^{(2)} - \frac{i}{2g}} \,, \qquad j = 1, \ldots, m_1^{(2)} , \tag{5.11}
$$

$$
\prod_{j=1}^{m_1^{(\alpha)}} \frac{\tilde{\mu}_l^{(\alpha)} - x^+(\lambda_j^{(\alpha)}) - \frac{1}{x^+(\lambda_j^{(\alpha)})} + \frac{i}{2g}}{\tilde{\mu}_l^{(\alpha)} - x^+(\lambda_j^{(\alpha)}) - \frac{1}{x^+(\lambda_j^{(\alpha)})} - \frac{i}{2g}} = \prod_{\substack{k=1\\k\neq l}}^{m_2^{(\alpha)}} \frac{\tilde{\mu}_l^{(\alpha)} - \tilde{\mu}_k^{(\alpha)} + \frac{i}{g}}{\tilde{\mu}_l^{(\alpha)} - \tilde{\mu}_k^{(\alpha)} - \frac{i}{g}} \,, \qquad l = 1, \ldots, m_2^{(\alpha)} ; \ \alpha = 1, 2 , \tag{5.12}
$$

where $P = \sum_{k=1}^{N} p_k$ is the total momentum. Following the change in notation in MM (47)-(49), (51), so that

$$
N = K_4 \,, \quad m_1^{(1)} = K_1 + K_3 \,, \quad m_1^{(2)} = K_5 + K_7 \,, \quad m_2^{(1)} = K_2 \,, \quad m_2^{(2)} = K_6 \,, \tag{5.13}
$$

and the coefficients (5.8) are given by

$$
c_1 = e^{i\gamma_1 (K_4 - K_5 - K_7)/2} e^{i(\gamma_3 - \gamma_2) J/2} \,, \quad c_2 = e^{-i\gamma_1 (K_4 - K_1 - K_3)/2} e^{i(\gamma_3 + \gamma_2) J/2} \,, \tag{5.14}
$$

[11]Note that $m_j \mapsto m_j^{(1)}$, $\dot{m}_j \mapsto m_j^{(2)}$ for $j = 1, 2$.

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eqs. (5.10), (5.11) become (cf. MM (52))

$$c_1 e^{-iP/2} \prod_{i=1}^{K_4} \frac{1-\frac{g^2}{x_{4,i}^- x_{1,j}}}{1-\frac{g^2}{x_{4,i}^+ x_{1,j}}} = \prod_{l=1}^{K_2} \frac{u_{1,j}-u_{2,l}+\frac{i}{2}}{u_{1,j}-u_{2,l}-\frac{i}{2}}, \quad j=1,\ldots,K_1 \tag{5.15}$$

$$c_1 e^{iP/2} \prod_{i=1}^{K_4} \frac{x_{4,i}^- - x_{3,j}}{x_{4,i}^+ - x_{3,j}} = \prod_{l=1}^{K_2} \frac{u_{3,j}-u_{2,l}+\frac{i}{2}}{u_{3,j}-u_{2,l}-\frac{i}{2}}, \quad j=1,\ldots,K_3 \tag{5.16}$$

$$c_2 e^{iP/2} \prod_{i=1}^{K_4} \frac{x_{4,i}^- - x_{5,j}}{x_{4,i}^+ - x_{5,j}} = \prod_{l=1}^{K_6} \frac{u_{5,j}-u_{6,l}+\frac{i}{2}}{u_{5,j}-u_{6,l}-\frac{i}{2}}, \quad j=1,\ldots,K_5 \tag{5.17}$$

$$c_2 e^{-iP/2} \prod_{i=1}^{K_4} \frac{1-\frac{g^2}{x_{4,i}^- x_{7,j}}}{1-\frac{g^2}{x_{4,i}^+ x_{7,j}}} = \prod_{l=1}^{K_6} \frac{u_{7,j}-u_{6,l}+\frac{i}{2}}{u_{7,j}-u_{6,l}-\frac{i}{2}}, \quad j=1,\ldots,K_7 \tag{5.18}$$

The undeformed eqs. (5.12) become the same as MM (53), namely,

$$\prod_{j=1}^{K_1} \frac{u_{2,l}-u_{1,j}+\frac{i}{2}}{u_{2,l}-u_{1,j}-\frac{i}{2}} \prod_{j=1}^{K_3} \frac{u_{2,l}-u_{3,j}+\frac{i}{2}}{u_{2,l}-u_{3,j}-\frac{i}{2}} = \prod_{\substack{k=1\\k\neq l}}^{K_2} \frac{u_{2,l}-u_{2,k}+i}{u_{2,l}-u_{2,k}-i}, \qquad l=1,\ldots,K_2$$
$$\prod_{j=1}^{K_5} \frac{u_{6,l}-u_{5,j}+\frac{i}{2}}{u_{6,l}-u_{5,j}-\frac{i}{2}} \prod_{j=1}^{K_7} \frac{u_{6,l}-u_{7,j}+\frac{i}{2}}{u_{6,l}-u_{7,j}-\frac{i}{2}} = \prod_{\substack{k=1\\k\neq l}}^{K_6} \frac{u_{6,l}-u_{6,k}+i}{u_{6,l}-u_{6,k}-i}, \qquad l=1,\ldots,K_6 \tag{5.19}$$

Finally, we consider the equations for the type-4 Bethe roots (i.e., corresponding to the middle node of the Dynkin diagram). These come from the Bethe-Yang equations (cf. (3.36))

$$\begin{aligned} e^{-ip_k L} &= \tilde{\Lambda}(p_k) \\ &= c_1 c_2 \prod_{i=1}^{N} \left[S_0(p_k,p_i) \frac{x^-(p_i)-x^+(p_k)}{x^+(p_i)-x^-(p_k)} \frac{\eta(p_i)}{\eta(p_k)} \right]^2 \\ &\quad \times \prod_{j=1}^{m_1} \eta(p_k) \frac{x^-(p_k)-x^+(\lambda_j)}{x^+(p_k)-x^+(\lambda_j)} \prod_{j=1}^{\dot{m}_1} \eta(p_k) \frac{x^-(p_k)-x^+(\dot{\lambda}_j)}{x^+(p_k)-x^+(\dot{\lambda}_j)}, \end{aligned} \tag{5.20}$$

where we have used our result (5.7) for $\Lambda(\lambda)$. Setting

$$L = -J \tag{5.21}$$

as proposed by MM, substituting the result for the scalar factor from MM (36), and changing notations as above, we obtain (cf. MM (50))

$$e^{ip_k\left[J+K_4-\frac{1}{2}(K_3-K_1)-\frac{1}{2}(K_5-K_7)\right]} = c_1 c_2 e^{iP} \prod_{\substack{i=1\\ i\neq k}}^{K_4} \left[\frac{x_{4,k}^+ - x_{4,i}^-}{x_{4,k}^- - x_{4,i}^+}\right] \left[\frac{1-\frac{g^2}{x_{4,k}^+ x_{4,i}^-}}{1-\frac{g^2}{x_{4,k}^- x_{4,i}^+}}\right] [\sigma(p_k,p_i)]^2$$
$$\times \prod_{j=1}^{K_3} \frac{x_{4,k}^- - x_{3,j}}{x_{4,k}^+ - x_{3,j}} \prod_{j=1}^{K_1} \frac{1-\frac{g^2}{x_{4,k}^- x_{1,j}}}{1-\frac{g^2}{x_{4,k}^+ x_{1,j}}}$$
$$\times \prod_{j=1}^{K_5} \frac{x_{4,k}^- - x_{5,j}}{x_{4,k}^+ - x_{5,j}} \prod_{j=1}^{K_7} \frac{1-\frac{g^2}{x_{4,k}^- x_{7,j}}}{1-\frac{g^2}{x_{4,k}^+ x_{7,j}}}, \quad k=1,\ldots,K_4 . \quad (5.22)$$

For the undeformed case $\gamma_i = 0$, one recovers the corresponding Bethe equations of Beisert and Staudacher [18] by setting $P = 0$ and recalling the result for the angular momentum charge

$$J = \mathcal{L} - K_4 + \frac{1}{2}(K_3 - K_1) + \frac{1}{2}(K_5 - K_7)\,, \quad (5.23)$$

where we denote by $\mathcal{L}$ the parameter used in [9, 18] to identify the length of the chain.

The twisted transfer matrix eigenvalue (5.7), which we have obtained from diagonalizing the twisted transfer matrix (5.5) based on the twisted scattering matrix (5.2) and the twisted boundary condition (5.6), is equivalent to the transfer matrix eigenvalue proposed in [16]; and for the special case of β-deformation, it is equivalent to the result in [17].

6 Comparison with BR

The paper [9] of Beisert and Roiban, to which we refer by BR, proposes a three-parameter $(\gamma_1\,, \gamma_2\,, \gamma_3)$ deformation of the all-loop asymptotic Bethe equations of Beisert and Staudacher [18]. The β-deformation [4] corresponds to a special case with $\mathcal{N} = 1$ supersymmetry,

$$\gamma_1 = \gamma_2 = \gamma_3 = 2\pi\beta\,. \quad (6.1)$$

The deformed all-loop Bethe equations are given by BR (5.39)

$$e^{i(\mathbf{AK})_0}\, U_0 = 1, \qquad e^{i(\mathbf{AK})_j}\, U_j(x_{j,k}) \prod_{\substack{j'=1\\ (j',k')\neq(j,k)}}^{7} \prod_{k'=1}^{K_{j'}} \frac{u_{j,k} - u_{j',k'} + \frac{i}{2}M_{j,j'}}{u_{j,k} - u_{j',k'} - \frac{i}{2}M_{j,j'}} = 1\,, \quad (6.2)$$

where

$$U_0 = \prod_{k=1}^{K_4} \frac{x_{4,k}^+}{x_{4,k}^-}\,, \qquad U_1(x) = U_3^{-1}(x) = U_5^{-1}(x) = U_7(x) = \prod_{k=1}^{K_4} S_{\mathrm{aux}}(x_{4,k}, x) \quad (6.3)$$

Figure 1. Dynkin diagram of $su(2,2|4)$.

and

$$U_4(x) = U_{\rm s}(x)\left(\frac{x^-}{x^+}\right)^{\mathcal{L}} \prod_{k=1}^{K_1} S_{\rm aux}^{-1}(x,x_{1,k}) \prod_{k=1}^{K_3} S_{\rm aux}(x,x_{3,k}) \prod_{k=1}^{K_5} S_{\rm aux}(x,x_{5,k}) \prod_{k=1}^{K_7} S_{\rm aux}^{-1}(x,x_{7,k}). \tag{6.4}$$

Moreover,

$$S_{\rm aux}(x_1,x_2) = \frac{1-g^2/x_1^+ x_2}{1-g^2/x_1^- x_2}\,, \qquad U_{\rm s}(x) = \prod_{k=1}^{K_4} \sigma(x,x_{4,k})^2\,. \tag{6.5}$$

For the "$su(2)$" grading with $\eta_1 = \eta_2 = +1$ which we consider here, $M_{j,j'}$ is the Cartan matrix specified by figure 1 (see eq. (5.1) in [18]), and the twist matrix $\mathbf{A}$ is given[12] by BR (5.24). It then follows from BR (4.27) that

$$\begin{aligned}
(\mathbf{AK})_0 &= \frac{1}{2}\big[\gamma_2\,(-K_1-K_3+K_5+K_7) \\
&\qquad +\gamma_3\,(K_1+K_3-2K_4+K_5+K_7)\big]\,, && (6.7)\\
(\mathbf{AK})_2 &= (\mathbf{AK})_6 = 0\,, && (6.8)\\
(\mathbf{AK})_1 + \frac{1}{2}(\mathbf{AK})_0 &= -\frac{1}{2}\left[(\gamma_3-\gamma_2)J + \gamma_1\,(K_4-K_5-K_7)\right]\,, && (6.9)\\
(\mathbf{AK})_3 - \frac{1}{2}(\mathbf{AK})_0 &= (\mathbf{AK})_1 + \frac{1}{2}(\mathbf{AK})_0\,, && (6.10)\\
(\mathbf{AK})_5 - \frac{1}{2}(\mathbf{AK})_0 &= -\frac{1}{2}\left[(\gamma_3+\gamma_2)J - \gamma_1\,(K_4-K_1-K_3)\right]\,, && (6.11)\\
(\mathbf{AK})_7 + \frac{1}{2}(\mathbf{AK})_0 &= (\mathbf{AK})_5 - \frac{1}{2}(\mathbf{AK})_0\,, && (6.12)\\
(\mathbf{AK})_4 + (\mathbf{AK})_0 &= \gamma_3 J - \frac{1}{2}\gamma_1\,(K_5+K_7-K_1-K_3)\,. && (6.13)
\end{aligned}$$

Note that eqs. (6.2) and (6.3) imply that the total momentum P is given by

$$P = -(\mathbf{AK})_0\,. \tag{6.14}$$

We now compare the BR Bethe equations with the ones which we derived in the previous section.

[12] This matrix is given in terms of the parameters $(\delta_1,\delta_2,\delta_3)$, which are related to $(\gamma_1\,,\gamma_2\,,\gamma_3)$ through eqs. BR (5.2) and BR (5.3); i.e.,

$$\gamma_1 = -\delta_1 - 2\delta_2 - \delta_3\,, \quad \gamma_2 = -\delta_1 - \delta_3\,, \quad \gamma_3 = -\delta_1 + \delta_3\,. \tag{6.6}$$

The fact that eqs. (5.19) are not deformed matches with (6.8) and (6.2) with $j = 2, 6$.

In order to facilitate the further comparisons, let us rewrite eqs. (5.15)–(5.18) in the form (6.2):

$$c_1^{-1} e^{iP/2} \prod_{i=1}^{K_4} \frac{1 - \frac{g^2}{x_{4,i}^+ x_{1,j}}}{1 - \frac{g^2}{x_{4,i}^- x_{1,j}}} \prod_{l=1}^{K_2} \frac{u_{1,j} - u_{2,l} + \frac{i}{2}}{u_{1,j} - u_{2,l} - \frac{i}{2}} = 1, \quad j = 1, \dots, K_1 \,, \tag{6.15}$$

$$c_1^{-1} e^{-iP/2} \prod_{i=1}^{K_4} \frac{x_{4,i}^+ - x_{3,j}}{x_{4,i}^- - x_{3,j}} \prod_{l=1}^{K_2} \frac{u_{3,j} - u_{2,l} + \frac{i}{2}}{u_{3,j} - u_{2,l} - \frac{i}{2}} = 1, \quad j = 1, \dots, K_3 \,, \tag{6.16}$$

$$c_2^{-1} e^{-iP/2} \prod_{i=1}^{K_4} \frac{x_{4,i}^+ - x_{5,j}}{x_{4,i}^- - x_{5,j}} \prod_{l=1}^{K_6} \frac{u_{5,j} - u_{6,l} + \frac{i}{2}}{u_{5,j} - u_{6,l} - \frac{i}{2}} = 1, \quad j = 1, \dots, K_5 \,, \tag{6.17}$$

$$c_2^{-1} e^{iP/2} \prod_{i=1}^{K_4} \frac{1 - \frac{g^2}{x_{4,i}^+ x_{7,j}}}{1 - \frac{g^2}{x_{4,i}^- x_{7,j}}} \prod_{l=1}^{K_6} \frac{u_{7,j} - u_{6,l} + \frac{i}{2}}{u_{7,j} - u_{6,l} - \frac{i}{2}} = 1, \quad j = 1, \dots, K_7 \,. \tag{6.18}$$

Substituting for P using (6.14), and noting the identities (proved using (5.14), (5.23) and (6.7)–(6.12)),

$$\begin{aligned} c_1^{-1} e^{iP/2} &= e^{i(\mathbf{AK})_1} \,, \quad c_1^{-1} e^{-iP/2} = e^{i(\mathbf{AK})_3} \,, \\ c_2^{-1} e^{-iP/2} &= e^{i(\mathbf{AK})_5} \,, \quad c_2^{-1} e^{iP/2} = e^{i(\mathbf{AK})_7} \,, \end{aligned} \tag{6.19}$$

we see that eqs. (6.15)–(6.18) match with (6.2) with $j = 1, 3, 5, 7$ respectively. Moreover, the identity

$$c_1 c_2 e^{iP} = e^{i(\mathbf{AK})_4} \tag{6.20}$$

implies that (5.22) matches with (6.2) with $j = 4$. In summary, provided we take P as in (6.14), the Bethe equations of section 5 match with those in BR.

7 Discussion

We have shown that the Beisert-Roiban Bethe equations (6.2) for the 3-parameter deformation of AdS_5/CFT_4 can be derived from the S-matrix with the Drinfeld-Reshetikhin twist (5.2)–(5.4), together with the c-number twist (5.6) of the boundary conditions. This result places the twisted Bethe equations on a firmer conceptual footing. Our result also reproduces the proposed twisted transfer matrix eigenvalue of [16]. As explained in appendix C, this result also justifies the deformed S-matrix elements used in [15] to compute the anomalous dimension of the Konishi operator in β-deformed $\mathcal{N} = 4$ SYM via the Lüscher formula. Indeed, we can recover with our approach the Lüscher correction for all known cases in the literature, both $su(2)$ and $sl(2)$.

We demonstrate in appendix D that the transfer matrix is spectrally equivalent to a transfer matrix which is constructed using instead *untwisted* S-matrices and boundary conditions with *operatorial* twists. It is the latter type of transfer matrix which is considered in [17]. A similar spectral equivalence was noted for the β-deformed $su(2)$ sector at one loop in [5]. Finally, in appendix E we transform our twisted Bethe ansatz results from the "$su(2)$" grading to the "$sl(2)$" grading, and show that the results agree with both [9] and [17].

The scattering matrix is a fundamental object in integrable systems and can be checked by various means. As its semiclassical limit corresponds to time delays, it would be nice to check our proposal against a classical string theory calculation based directly on the Lunin-Maldacena [6] background.

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A Derivation of the exchange relations

An important ingredient of the algebraic Bethe ansatz approach is the set of exchange relations which are obeyed by the matrix elements of the monodromy matrix. Here we explain how to derive the exchange relations used in this paper.

A.1 $su(2)$ principal chiral model

The exchange relations for the spin-1/2 XXX quantum spin chain are well known (see, for example, [50, 51]). We shall need the relations between the diagonal operators A, D and the creation operator B

$$A(u)\,B(v) = \frac{u-v-i}{u-v}B(v)\,A(u) + \frac{i}{u-v}B(u)\,A(v)\,, \tag{A.1}$$

$$D(u)\,B(v) = \frac{u-v+i}{u-v}B(v)\,D(u) - \frac{i}{u-v}B(u)\,D(v)\,, \tag{A.2}$$

as well as relations among the diagonal operators,

$$[A(u)\,,A(v)] = [D(u)\,,D(v)] = 0\,, \tag{A.3}$$

$$D(u)\,A(v) = A(v)\,D(u) + \frac{i}{u-v}\left(B(v)\,C(u) - B(u)\,C(v)\right)\,. \tag{A.4}$$

The same relations hold for the corresponding dotted operators $\dot{A}, \dot{B}, \dot{C}, \dot{D}$; and the dotted and undotted operators commute with each other.

The exchange relations between the diagonal operators $\mathcal{T}_{jj}$ and the creation operators $\mathcal{T}_{12}, \mathcal{T}_{13}$ can be classified into two types, depending on whether they make use of (A.3) or (A.4). The former are very simple, and are given in (3.18); the latter are more complicated, and are given in (3.19).

The first exchange relation in (3.18) can be easily derived using (A.1) and (A.3):

$$\begin{aligned}
\mathcal{T}_{11}(u)\,\mathcal{T}_{13}(v) &= A(u)\,\dot{A}(u)\,B(v)\,\dot{A}(v) \\
&= A(u)\,B(v)\,\dot{A}(u)\,\dot{A}(v) \\
&= \left[\frac{u-v-i}{u-v}B(v)\,A(u) + \frac{i}{u-v}B(u)\,A(v)\right]\dot{A}(u)\,\dot{A}(v) \\
&= \frac{u-v-i}{u-v}\mathcal{T}_{13}(v)\,\mathcal{T}_{11}(u) + \frac{i}{u-v}\mathcal{T}_{13}(u)\,\mathcal{T}_{11}(v)\,,
\end{aligned} \tag{A.5}$$

and the remaining relations in (3.18) can be derived in a similar way.

The derivation of the first exchange relation in (3.19) also begins in a similar way:

$$\begin{aligned}
\mathcal{T}_{22}(u)\,\mathcal{T}_{13}(v) &= A(u)\,\dot{D}(u)\,B(v)\,\dot{A}(v) \\
&= A(u)\,B(v)\,\dot{D}(u)\,\dot{A}(v) \\
&= \left[\frac{u-v-i}{u-v}B(v)\,A(u) + \frac{i}{u-v}B(u)\,A(v)\right]\dot{D}(u)\,\dot{A}(v) \\
&= \frac{u-v-i}{u-v}B(v)\,A(u)\dot{D}(u)\,\dot{A}(v) + \frac{i}{u-v}\mathcal{T}_{24}(u)\,\mathcal{T}_{11}(v)\,.
\end{aligned} \tag{A.6}$$

We next observe that the first term on the r.h.s. can be re-expressed using (A.4) as follows

$$\begin{aligned}
&\frac{u-v-i}{u-v}B(v)\,A(u)\dot{D}(u)\,\dot{A}(v) \\
&\quad = \frac{u-v-i}{u-v}B(v)\,A(u)\left[\dot{A}(v)\,\dot{D}(u) + \frac{i}{u-v}\dot{B}(v)\,\dot{C}(u) - \frac{i}{u-v}\dot{B}(u)\,\dot{C}(v)\right] \\
&\quad = \frac{u-v-i}{u-v}\mathcal{T}_{13}(v)\,\mathcal{T}_{22}(u) + \frac{i(u-v-i)}{(u-v)^2}\mathcal{T}_{14}(v)\,\mathcal{T}_{21}(u) \\
&\qquad - \frac{i(u-v-i)}{(u-v)^2}B(v)\,A(u)\,\dot{B}(u)\,\dot{C}(v)\,.
\end{aligned} \tag{A.7}$$

Finally, we use again (A.1) to re-write the final term in (A.7) as

$$\begin{aligned}
&-\frac{i(u-v-i)}{(u-v)^2}B(v)\,A(u)\,\dot{B}(u)\,\dot{C}(v) \\
&\quad = -\frac{i(u-v-i)}{(u-v)^2}\left[\frac{u-v}{u-v-i}A(u)\,B(v) - \frac{i}{u-v-i}B(u)\,A(v)\right]\dot{B}(u)\,\dot{C}(v) \\
&\quad = -\frac{i}{u-v}\mathcal{T}_{12}(u)\,\mathcal{T}_{23}(v) - \frac{1}{(u-v)^2}\mathcal{T}_{14}(u)\,\mathcal{T}_{21}(v)\,.
\end{aligned} \tag{A.8}$$

Combining the results (A.6)–(A.8), we arrive at the first exchange relation in (3.19). The remaining relations in (3.19) can be derived in a similar way.

We remark that one can generate many other ("bad") exchange relations which differ from those given in (3.19). What singles out those in (3.19) is that the diagonal term gives the "wanted" contribution, while the rest of the terms give "unwanted" contributions. To get this right, it helps to know the desired final result, which was first found by other means in Sec 3.1. A further useful check is that these exchange relations have simple deformations (3.25), which does not seem to be the case for "bad" exchange relations.

A.2 Two copies of the Hubbard model

The exchange relations between the diagonal operators B, A_{jj}, D and the creation operators B_j are given in MR (34)-(36). For example,

$$B(\lambda)\, B_j(\mu) = \frac{i\alpha_2(\mu,\lambda)}{\alpha_9(\mu,\lambda)} B_j(\mu)\, B(\lambda) - \frac{i\alpha_5(\mu,\lambda)}{\alpha_9(\mu,\lambda)} B_j(\lambda)\, B(\mu)\,, \qquad j=1,2, \tag{A.9}$$

$$A_{jj}(\lambda)\, B_j(\mu) = -\frac{i\alpha_1(\lambda,\mu)}{\alpha_9(\lambda,\mu)} B_j(\mu)\, A_{jj}(\lambda) + \frac{i\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)} B_j(\lambda)\, A_{jj}(\mu)\,, \quad j=1,2, \tag{A.10}$$

where there is *no* sum over repeated indices. We shall also need exchange relations among the diagonal operators (see (12.D.1) in [52]),

$$B(\lambda)\, B(\mu) = B(\mu)\, B(\lambda)\,, \tag{A.11}$$

$$A_{jj}(\lambda)\, B(\mu) = B(\mu)\, A_{jj}(\lambda) - \frac{i\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)} \left(B_j(\mu)\, C_j(\lambda) - B_j(\lambda)\, C_j(\mu)\right)\,, \quad j=1,2, \tag{A.12}$$

$$A_{jj}(\lambda)\, D(\mu) = D(\mu)\, A_{jj}(\lambda) - \frac{i\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)} \left(C_j^*(\mu)\, B_j^*(\lambda) - C_j^*(\lambda)\, B_j^*(\mu)\right)\,, \quad j=1,2, \tag{A.13}$$

$$\begin{aligned} D(\lambda)\, B(\mu) = B(\mu)\, D(\lambda) &- \frac{\alpha_4(\lambda,\mu)}{\alpha_7(\lambda,\mu)} \left(F(\lambda)\, C(\mu) - F(\mu)\, C(\lambda)\right) \\ &- \frac{i\alpha_{10}(\lambda,\mu)}{\alpha_7(\lambda,\mu)} \left(B_1^*(\lambda)\, C_2(\mu) - B_2^*(\lambda)\, C_1(\mu) + B_1(\mu)\, C_2^*(\lambda) - B_2(\mu)\, C_1^*(\lambda)\right)\,. \end{aligned} \tag{A.14}$$

The same relations hold for the corresponding dotted operators.

The exchange relations between the diagonal operators $\mathcal{T}_{j,j}$ and the creation operators $\mathcal{T}_{1,2}, \mathcal{T}_{1,3}, \mathcal{T}_{1,5}, \mathcal{T}_{1,9}$ can be classified into four types, depending on which of the four relations (A.11)–(A.14) they make use of. The relations of the first type which make use of (A.11) are the simplest; while the relations of the fourth type which make use of (A.14) are the most complicated. All of these relations can be derived using the same procedure which we used for the principal chiral model.

Here is an example of the first type:

$$\begin{aligned} \mathcal{T}_{1,1}(\lambda)\, \mathcal{T}_{1,2}(\mu) &= B(\lambda)\, \dot{B}(\lambda)\, B(\mu)\, \dot{B}_1(\mu) \\ &= B(\lambda)\, B(\mu)\, \dot{B}(\lambda)\, \dot{B}_1(\mu) \\ &= B(\lambda)\, B(\mu) \left[\frac{i\alpha_2(\mu,\lambda)}{\alpha_9(\mu,\lambda)} \dot{B}_1(\mu)\, \dot{B}(\lambda) - \frac{i\alpha_5(\mu,\lambda)}{\alpha_9(\mu,\lambda)} \dot{B}_1(\lambda)\, \dot{B}(\mu)\right] \\ &= \frac{i\alpha_2(\mu,\lambda)}{\alpha_9(\mu,\lambda)} \mathcal{T}_{1,2}(\mu)\, \mathcal{T}_{1,1}(\lambda) - \frac{i\alpha_5(\mu,\lambda)}{\alpha_9(\mu,\lambda)} \mathcal{T}_{1,2}(\lambda)\, \mathcal{T}_{1,1}(\mu)\,, \end{aligned} \tag{A.15}$$

where we have used (A.9) to pass to the third line.

For an example of the second type, let us consider

$$\begin{aligned}\mathcal{T}_{2,2}(\lambda)\,\mathcal{T}_{1,5}(\mu) &= B(\lambda)\,\dot{A}_{11}(\lambda)\,B_1(\mu)\,\dot{B}(\mu)\\ &= B(\lambda)\,B_1(\mu)\,\dot{A}_{11}(\lambda)\,\dot{B}(\mu)\\ &= \left[\frac{i\alpha_2(\mu,\lambda)}{\alpha_9(\mu,\lambda)}B_1(\mu)\,B(\lambda)-\frac{i\alpha_5(\mu,\lambda)}{\alpha_9(\mu,\lambda)}B_1(\lambda)\,B(\mu)\right]\dot{A}_{11}(\lambda)\,\dot{B}(\mu)\\ &= \frac{i\alpha_2(\mu,\lambda)}{\alpha_9(\mu,\lambda)}B_1(\mu)\,B(\lambda)\,\dot{A}_{11}(\lambda)\,\dot{B}(\mu)-\frac{i\alpha_5(\mu,\lambda)}{\alpha_9(\mu,\lambda)}\mathcal{T}_{2,6}(\lambda)\,\mathcal{T}_{1,1}(\mu)\,.\end{aligned}\tag{A.16}$$

The first term on the r.h.s. can be re-expressed using (A.12) as follows

$$\begin{aligned}&\frac{i\alpha_2(\mu,\lambda)}{\alpha_9(\mu,\lambda)}B_1(\mu)\,B(\lambda)\,\dot{A}_{11}(\lambda)\,\dot{B}(\mu)\\ &\quad= \frac{i\alpha_2(\mu,\lambda)}{\alpha_9(\mu,\lambda)}B_1(\mu)\,B(\lambda)\left[\dot{B}(\mu)\,\dot{A}_{11}(\lambda)-\frac{i\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)}\left(\dot{B}_1(\mu)\,\dot{C}_1(\lambda)-\dot{B}_1(\lambda)\,\dot{C}_1(\mu)\right)\right]\\ &\quad= \frac{i\alpha_2(\mu,\lambda)}{\alpha_9(\mu,\lambda)}\mathcal{T}_{1,5}(\mu)\,\mathcal{T}_{2,2}(\lambda)+\frac{\alpha_2(\mu,\lambda)\alpha_5(\lambda,\mu)}{\alpha_9(\mu,\lambda)\alpha_9(\lambda,\mu)}\mathcal{T}_{1,6}(\mu)\,\mathcal{T}_{2,1}(\lambda)\\ &\qquad-\frac{\alpha_2(\mu,\lambda)\alpha_5(\lambda,\mu)}{\alpha_9(\mu,\lambda)\alpha_9(\lambda,\mu)}B_1(\mu)\,B(\lambda)\,\dot{B}_1(\lambda)\,\dot{C}_1(\mu)\,.\end{aligned}\tag{A.17}$$

Finally, we use again (A.9) to re-write the final term in (A.17) as

$$\begin{aligned}&-\frac{\alpha_2(\mu,\lambda)\alpha_5(\lambda,\mu)}{\alpha_9(\mu,\lambda)\alpha_9(\lambda,\mu)}B_1(\mu)\,B(\lambda)\,\dot{B}_1(\lambda)\,\dot{C}_1(\mu)\\ &\quad= \frac{i\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)}\left[B(\lambda)\,B_1(\mu)+i\frac{\alpha_5(\mu,\lambda)}{\alpha_9(\mu,\lambda)}B_1(\lambda)\,B(\mu)\right]\dot{B}_1(\lambda)\,\dot{C}_1(\mu)\\ &\quad= \frac{i\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)}\mathcal{T}_{1,2}(\lambda)\,\mathcal{T}_{2,5}(\mu)-\frac{\alpha_5(\mu,\lambda)\alpha_5(\lambda,\mu)}{\alpha_9(\mu,\lambda)\alpha_9(\lambda,\mu)}\mathcal{T}_{1,6}(\lambda)\,\mathcal{T}_{2,1}(\mu)\end{aligned}\tag{A.18}$$

Combining the results (A.16)–(A.18), we arrive at the exchange relation

$$\begin{aligned}\mathcal{T}_{2,2}(\lambda)\,\mathcal{T}_{1,5}(\mu) &= \frac{i\alpha_2(\mu,\lambda)}{\alpha_9(\mu,\lambda)}\mathcal{T}_{1,5}(\mu)\,\mathcal{T}_{2,2}(\lambda)+\frac{\alpha_2(\mu,\lambda)\alpha_5(\lambda,\mu)}{\alpha_9(\mu,\lambda)\alpha_9(\lambda,\mu)}\mathcal{T}_{1,6}(\mu)\,\mathcal{T}_{2,1}(\lambda)\\ &\quad+\frac{i\alpha_5(\mu,\lambda)}{\alpha_9(\mu,\lambda)}\mathcal{T}_{2,6}(\lambda)\,\mathcal{T}_{1,1}(\mu)+\frac{i\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)}\mathcal{T}_{1,2}(\lambda)\,\mathcal{T}_{2,5}(\mu)\\ &\quad-\frac{\alpha_5(\mu,\lambda)\alpha_5(\lambda,\mu)}{\alpha_9(\mu,\lambda)\alpha_9(\lambda,\mu)}\mathcal{T}_{1,6}(\lambda)\,\mathcal{T}_{2,1}(\mu)\,.\end{aligned}\tag{A.19}$$

An example of an exchange relation of the fourth type, which can be derived in a similar manner using (A.10) and (A.14), is

$$\begin{aligned}&\mathcal{T}_{8,8}(\lambda)\,\mathcal{T}_{1,5}(\mu) = -\frac{i\alpha_1(\lambda,\mu)}{\alpha_9(\lambda,\mu)}\mathcal{T}_{1,5}(\mu)\,\mathcal{T}_{8,8}(\lambda)+\frac{i\alpha_5(\lambda,\mu)}{\alpha_9(\lambda,\mu)}\mathcal{T}_{4,8}(\lambda)\,\mathcal{T}_{5,5}(\mu)\\ &\quad-\frac{i\alpha_1(\lambda,\mu)\alpha_4(\lambda,\mu)}{\alpha_9(\lambda,\mu)\alpha_7(\lambda,\mu)}\mathcal{T}_{1,8}(\mu)\,\mathcal{T}_{8,5}(\lambda)-\frac{\alpha_1(\lambda,\mu)\alpha_{10}(\lambda,\mu)}{\alpha_9(\lambda,\mu)\alpha_7(\lambda,\mu)}\left(\mathcal{T}_{1,6}(\mu)\,\mathcal{T}_{8,7}(\lambda)-\mathcal{T}_{1,7}(\mu)\,\mathcal{T}_{8,6}(\lambda)\right)\\ &\quad-\frac{\alpha_4(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\mathcal{T}_{5,8}(\lambda)\,\mathcal{T}_{4,5}(\mu)-\frac{i\alpha_{10}(\lambda,\mu)}{\alpha_7(\lambda,\mu)}\left(\mathcal{T}_{6,8}(\lambda)\,\mathcal{T}_{3,5}(\mu)-\mathcal{T}_{7,8}(\lambda)\,\mathcal{T}_{2,5}(\mu)\right)\\ &\quad+\frac{i\alpha_5(\lambda,\mu)\alpha_4(\lambda,\mu)}{\alpha_9(\lambda,\mu)\alpha_7(\lambda,\mu)}\mathcal{T}_{1,8}(\lambda)\,\mathcal{T}_{8,5}(\mu)+\frac{\alpha_5(\lambda,\mu)\alpha_{10}(\lambda,\mu)}{\alpha_9(\lambda,\mu)\alpha_7(\lambda,\mu)}\left(\mathcal{T}_{2,8}(\lambda)\,\mathcal{T}_{7,5}(\mu)-\mathcal{T}_{3,8}(\lambda)\,\mathcal{T}_{6,5}(\mu)\right).\end{aligned}\tag{A.20}$$

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It is not feasible to present all 64 exchange relations in their entirety. Nevertheless, the "wanted" (diagonal) terms of all these exchange relations are given in (4.16).

B Crossing symmetry

We verify here that the twisted S-matrix (5.2) has the standard crossing symmetry property. For a single copy of the untwisted $su(2|2)$ S-matrix in the elliptic parametrization [25], the crossing relation [23] is given by

$$C_1^{-1} S_{12}^{t_1}(z_1, z_2) C_1 S_{12}(z_1 + \omega_2, z_2) = I_{12} \,, \quad C_1^{-1} S_{12}^{t_1}(z_1, z_2) C_1 S_{12}(z_1, z_2 - \omega_2) = I_{12} \,, \tag{B.1}$$

where C is the 4×4 matrix given by

$$C = \begin{pmatrix} \sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} , \tag{B.2}$$

and σ_2 is the second Pauli matrix.

We write the full untwisted $su(2|2)^2$ S-matrix (5.1) as

$$\mathcal{S}_{1234}(z_1, z_2) = S_{13}(z_1, z_2)\, S_{24}(z_1, z_2) \,. \tag{B.3}$$

One can easily check that it obeys the crossing relation

$$C_1^{-1} C_2^{-1} \mathcal{S}_{1234}^{t_1 t_2}(z_1, z_2) C_1 C_2 \mathcal{S}_{1234}(z_1 + \omega_2, z_2) = I_{1234} \,, \tag{B.4}$$

and a similar second relation. We write the twisted S-matrix (5.2) as

$$\tilde{\mathcal{S}}_{1234}(z_1, z_2) = F_{1234}\, \mathcal{S}_{1234}(z_1, z_2)\, F_{1234} \,, \tag{B.5}$$

It should obey the same crossing relation, i.e.,

$$C_1^{-1} C_2^{-1} \tilde{\mathcal{S}}_{1234}^{t_1 t_2}(z_1, z_2) C_1 C_2 \tilde{\mathcal{S}}_{1234}(z_1 + \omega_2, z_2) = I_{1234} \,. \tag{B.6}$$

We find that the twist matrix F (5.3) obeys the relation

$$C_1^{-1} C_2^{-1} F_{1234} C_1 C_2 = F_{1234}^{-1} \,. \tag{B.7}$$

Using this identity, one can check that the crossing relation (B.6) is indeed satisfied.

C Lüscher correction

We show here that the proposed twisted scattering matrix and twisted boundary condition reproduce the wrapping correction not only for the Konishi operator [15] but also for generic multiparticle states both in the $su(2)$ and in the $sl(2)$ sectors analyzed in [16, 17].

Let us start with the $su(2)$ sector. It consists of identical particles carrying the labels $1\dot{1} = X$. In the Lüscher correction, we need the scattering matrix of the X particle on the mirror boundstates. Since only the R_1 charge gives a nonvanishing contribution on X, the twist factor of the scattering matrix will be

$$F = e^{i\gamma_{12} \mathbb{I} \otimes R_2} = q^{\mathbb{I} \otimes R_2} \,, \tag{C.1}$$

where in the last equality we focus on the β-deformation only: $\gamma_{12} = \frac{1}{2}\gamma = \pi\beta$, and $q = e^{i\pi\beta}$. Evaluating R_2 on the mirror boundstates gives the following twist of the S-matrix elements:

	$\vert B_k\rangle_I$	$\vert B_k\rangle_{II}$	$\vert F_k\rangle_I$	$\vert F_k\rangle_{II}$
1	1	1	q	q^{-1}

	$\vert \dot{B}_k\rangle_I$	$\vert \dot{B}_k\rangle_{II}$	$\vert \dot{F}_k\rangle_I$	$\vert \dot{F}_k\rangle_{II}$
$\dot{1}$	1	1	q^{-1}	q

Taking into account that the scattering is diagonal in this sector, the wrapping correction for N particles of type $1\dot{1}$ will contain the N^{th} power of the above expressions.

Additionally, we also have to remember that the mirror boundstate should satisfy the twisted boundary condition, which for $\gamma_{23} = \gamma_{13} = \frac{1}{2}\gamma$ reads as: $q^{4J\,(h\otimes 1)}$. In detail, we have

	$\vert B_k\rangle_I$	$\vert B_k\rangle_{II}$	$\vert F_k\rangle_I$	$\vert F_k\rangle_{II}$
BC	1	1	1	1

	$\vert \dot{B}_k\rangle_I$	$\vert \dot{B}_k\rangle_{II}$	$\vert \dot{F}_k\rangle_I$	$\vert \dot{F}_k\rangle_{II}$
BC	1	1	q^{2J}	q^{-2J}

Combining the two results, we can equivalently describe our deformation with a different twisted boundary condition (BC') given by

	$\vert B_k\rangle_I$	$\vert B_k\rangle_{II}$	$\vert F_k\rangle_I$	$\vert F_k\rangle_{II}$
BC'	1	1	q^N	q^{-N}

	$\vert \dot{B}_k\rangle_I$	$\vert \dot{B}_k\rangle_{II}$	$\vert \dot{F}_k\rangle_I$	$\vert \dot{F}_k\rangle_{II}$
BC'	1	1	q^{2J-N}	q^{-2J+N}

which completely agrees with the $su(2)$ part of table 1 in [17]. As the scatterings in the $sl(2)$ sectors are not twisted, our twisted boundary conditions are equivalent to the $sl(2)$ part of table 1 in [17].

D Operatorial twists of the boundary conditions

We demonstrate here that our transfer matrix, which is constructed with twisted S-matrices and boundary conditions with c-number twists, is spectrally equivalent to a transfer matrix which is constructed with *untwisted* S-matrices and boundary conditions with *operatorial* twists. It is the latter type of transfer matrix which is considered in [17]. Moreover, we show directly that the same twisted Bethe equations can also be derived starting from the latter transfer matrix.

D.1 $su(2)$ principal chiral model

For the case of the $su(2)$ principal chiral model, the transfer matrix is given by (3.8). Let us now streamline the notation, and denote $a\dot{a}$ by A, and $j\dot{j}$ by j for $j = 1, \ldots, L$. The transfer matrix (3.8) then takes the form

$$\tilde{t}(u) = \mathrm{tr}_A\, M_A\, \tilde{\mathcal{T}}_A(u)\,, \qquad \tilde{\mathcal{T}}_A(u) = \prod_{j=1}^{L} \tilde{\mathcal{S}}_{Aj}(u)\,, \tag{D.1}$$

where

$$\tilde{\mathcal{S}}_{Aj}(u) = F_{Aj}\, \mathcal{S}_{Aj}(u)\, F_{Aj}\,. \tag{D.2}$$

The F-matrix satisfies [31]

$$F_{12}\, F_{13}\, F_{23} = F_{23}\, F_{13}\, F_{12}\,, \tag{D.3}$$

as well as

$$\mathcal{S}_{12}(u)\, F_{13}\, F_{23} = F_{23}\, F_{13}\, \mathcal{S}_{12}(u)\,. \tag{D.4}$$

This equation means that the twist appears as a seam (defect) in the spin chain, whose location can be changed without altering the spectrum [53]. To see this, we observe that under spectral equivalence ($\equiv$) the S-matrix and the F-matrix acting in different quantum spaces commute,

$$\mathcal{S}_{12}(u)\, F_{13} = F_{23}(F_{13}\, \mathcal{S}_{12}(u))\, F_{23}^{-1} \equiv F_{13}\, \mathcal{S}_{12}(u)\,. \tag{D.5}$$

The same is true for F-matrices,

$$F_{12}\, F_{13} = F_{23}(F_{13}\, F_{12})\, F_{23}^{-1} \equiv F_{13}\, F_{12}\,. \tag{D.6}$$

The transfer matrix (D.1) can therefore be written as

$$\begin{aligned} \tilde{t}(u) &= \mathrm{tr}_A\, M_A \prod_{j=1}^{L} F_{Aj}\, \mathcal{S}_{Aj}(u)\, F_{Aj} \\ &\equiv \mathrm{tr}_A\, M_A \prod_{j=1}^{L} F_{Aj}^2 \prod_{j=1}^{L} \mathcal{S}_{Aj}(u) \\ &= \mathrm{tr}_A\, \tilde{M}_A\, \mathcal{T}_A(u)\,, \end{aligned} \tag{D.7}$$

where

$$\tilde{M}_A = M_A \prod_{j=1}^{L} F_{Aj}^2\,, \qquad \mathcal{T}_A(u) = \prod_{j=1}^{L} \mathcal{S}_{Aj}(u)\,. \tag{D.8}$$

This shows that the transfer matrix (D.1) (which is constructed with the twisted S-matrices $\tilde{\mathcal{S}}$ and the matrix M_A which acts only on the auxiliary space) is spectrally equivalent to the transfer matrix (D.7) (which is constructed with the *untwisted* S-matrices $\mathcal{S}$ and the matrix $\tilde{M}_A$ which acts also on all the quantum spaces).[13]

Let us now explicitly evaluate $\tilde{M}_A$. The F-matrix (3.6) can be rewritten as

$$F_{Aj} = e^{i\gamma_1 (H_A \dot{H}_j - \dot{H}_A H_j)}\,, \tag{D.9}$$

where we have defined $H = h \otimes \mathbb{I}$ and $\dot{H} = \mathbb{I} \otimes h$. Hence,

$$\prod_{j=1}^{L} F_{Aj}^2 = e^{i2\gamma_1 \left[H_A \sum_{j=1}^{L} \dot{H}_j - \dot{H}_A \sum_{j=1}^{L} H_j\right]}\,. \tag{D.10}$$

Moreover, M_A (3.10) can be rewritten as

$$M_A = e^{i\gamma_2 H_A + i\gamma_3 \dot{H}_A}\,. \tag{D.11}$$

[13] A similar observation has been made by Foerster, Links and Roditi [36].

Hence, $\tilde{M}_A$ in (D.8) is given by

$$\begin{aligned}\tilde{M}_A &= e^{i\left(\gamma_2+2\gamma_1\sum_{j=1}^L \dot{H}_j\right)H_A+i\left(\gamma_3-2\gamma_1\sum_{j=1}^L H_j\right)\dot{H}_A} \\ &= e^{i\left(\gamma_2+2\gamma_1\dot{S}^z\right)h}\otimes e^{i(\gamma_3-2\gamma_1 S^z)h}\,,\end{aligned} \tag{D.12}$$

where the spin operators are given by

$$S^z = \sum_{j=1}^L H_j\,, \qquad \dot{S}^z = \sum_{j=1}^L \dot{H}_j\,. \tag{D.13}$$

Evidently, $\tilde{M}_A$ contains spin operators which act on the quantum space.

Finally, let us derive the Bethe equations corresponding to the transfer matrix (D.7). Using (D.12), we see that this transfer matrix is given by

$$\begin{aligned}\tilde{t}(u) = {}& e^{\frac{i}{2}\left[\gamma_2+\gamma_3+2\gamma_1\left(\dot{S}^z-S^z\right)\right]}\mathcal{T}_{11}(u)+e^{\frac{i}{2}\left[\gamma_2-\gamma_3+2\gamma_1\left(\dot{S}^z+S^z\right)\right]}\mathcal{T}_{22}(u) \\ &+e^{\frac{i}{2}\left[\gamma_3-\gamma_2-2\gamma_1\left(\dot{S}^z+S^z\right)\right]}\mathcal{T}_{33}(u)+e^{-\frac{i}{2}\left[\gamma_2+\gamma_3+2\gamma_1\left(\dot{S}^z-S^z\right)\right]}\mathcal{T}_{44}(u)\,.\end{aligned} \tag{D.14}$$

Recall that the spin operators satisfy the commutation relations

$$[S^z\,, B(u)] = -B(u)\,, \qquad [S^z\,, A(u)] = [S^z\,, D(u)] = 0\,, \tag{D.15}$$

and similarly for the operators with dots. Hence, the commutation relations of the spin operators with the creation operators are given by

$$\begin{aligned}[S^z\,, \mathcal{T}_{13}(u)] &= -\mathcal{T}_{13}(u)\,, \qquad & [S^z\,, \mathcal{T}_{12}(u)] &= 0 \\ \left[\dot{S}^z\,, \mathcal{T}_{13}(u)\right] &= 0\,, & \left[\dot{S}^z\,, \mathcal{T}_{12}(u)\right] &= -\mathcal{T}_{12}(u)\,.\end{aligned} \tag{D.16}$$

Moreover, acting on the pseudovacuum,

$$S^z(|0\rangle\otimes|\dot{0}\rangle) = \frac{L}{2}(|0\rangle\otimes|\dot{0}\rangle)\,, \qquad \dot{S}^z(|0\rangle\otimes|\dot{0}\rangle) = \frac{L}{2}(|0\rangle\otimes|\dot{0}\rangle)\,. \tag{D.17}$$

Therefore, acting on a general state (3.20),

$$e^{i\gamma S^z}|\Lambda\rangle = e^{i\gamma\left(\frac{L}{2}-m\right)}|\Lambda\rangle\,, \qquad e^{i\gamma\dot{S}^z}|\Lambda\rangle = e^{i\gamma\left(\frac{L}{2}-\dot{m}\right)}|\Lambda\rangle\,. \tag{D.18}$$

Acting with the transfer matrix (D.14) on a general state (3.20), we see (using also the untwisted exchange relations (3.18), (3.19) and the pseudovacuum eigenvalues (3.21)) that the corresponding eigenvalues are given by the same expression (3.29) which we obtained before. Hence, we arrive at the same twisted Bethe equations as before.

D.2 Two copies of the Hubbard model

For the case of two copies of the Hubbard model, the same argument as above implies that the transfer matrix (4.9) is spectrally equivalent to

$$\tilde{t}(\lambda) = \mathrm{str}_{a\dot{a}}\,\tilde{M}_{a\dot{a}}\mathcal{T}_{a\dot{a}}(\lambda)\,, \tag{D.19}$$

where the monodromy matrix $\mathcal{T}_{a\dot{a}}(\lambda)$ is not twisted, but the diagonal matrix $\tilde{M}$ contains spin-like operators which act on the quantum space,

$$\tilde{M} = e^{i(\gamma_2+\gamma_1\dot{\eta}^z)h} \otimes e^{i(\gamma_3-\gamma_1\eta^z)h}\,, \tag{D.20}$$

where the η^z operator is defined in MR (136). The matrix h is now given by (4.8).

The operator η^z has the following property given in MR (139)

$$\left[\eta^z\,, \vec{B}(\lambda)\right] = -\vec{B}(\lambda)\,. \tag{D.21}$$

Hence, the commutation relations of η^z and $\dot{\eta}^z$ with the creation operators are given by

$$\begin{aligned}&[\eta^z\,, \mathcal{T}_{1,2}(\lambda)] = [\eta^z\,, \mathcal{T}_{1,3}(\lambda)] = 0\,, \quad [\eta^z\,, \mathcal{T}_{1,5}(\lambda)] = -\mathcal{T}_{1,5}(\lambda)\,, \quad [\eta^z\,, \mathcal{T}_{1,9}(\lambda)] = -\mathcal{T}_{1,9}(\lambda)\,,\\ &[\dot{\eta}^z\,, \mathcal{T}_{1,2}(\lambda)] = -\mathcal{T}_{1,2}(\lambda)\,, \quad [\dot{\eta}^z\,, \mathcal{T}_{1,3}(\lambda)] = -\mathcal{T}_{1,3}(\lambda)\,, \quad [\dot{\eta}^z\,, \mathcal{T}_{1,5}(\lambda)] = [\dot{\eta}^z\,, \mathcal{T}_{1,9}(\lambda)] = 0\,.\end{aligned} \tag{D.22}$$

Moreover, acting on the pseudovacuum,

$$\eta^z(|0\rangle \otimes |\dot{0}\rangle) = L(|0\rangle \otimes |\dot{0}\rangle)\,, \qquad \dot{\eta}^z(|0\rangle \otimes |\dot{0}\rangle) = L(|0\rangle \otimes |\dot{0}\rangle)\,. \tag{D.23}$$

Therefore, acting on a general state,

$$e^{i\gamma\eta^z}|\Lambda\rangle = e^{i\gamma(L-n)}|\Lambda\rangle\,, \qquad e^{i\gamma\dot{\eta}^z}|\Lambda\rangle = e^{i\gamma(L-\dot{n})}|\Lambda\rangle\,. \tag{D.24}$$

Acting with the transfer matrix (D.19) on a general state, we find (using also the untwisted exchange relations (4.16) and the pseudovacuum eigenvalues (4.19)) that the corresponding eigenvalues are given by the same expression (4.31) which we obtained before. Hence, we arrive at the same twisted Bethe equations as before.

D.3 AdS/CFT

For the AdS/CFT case, it now follows that the transfer matrix (5.5) is spectrally equivalent to

$$\tilde{t}(\lambda) = \mathrm{str}_{a\dot{a}}\, \tilde{M}_{a\dot{a}} S_{a\dot{a}1\dot{1}}(\lambda, p_1) \ldots S_{a\dot{a}N\dot{N}}(\lambda, p_N)\,, \tag{D.25}$$

where the matrix $\tilde{M}_{a\dot{a}}$ is given by

$$\tilde{M} = e^{i[(\gamma_3-\gamma_2)J+\gamma_1\dot{\eta}^z]h} \otimes e^{i[(\gamma_3+\gamma_2)J-\gamma_1\eta^z]h}\,, \tag{D.26}$$

and h is given by (5.4). This leads to the same eigenvalues (5.7), and therefore the BR Bethe equations.

E $sl(2)$ grading

Here we transform our twisted Bethe ansatz results from the "$su(2)$" grading to the "$sl(2)$" grading, and show that the results agree with both [9] and [17].

We recall that the eigenvalues of the twisted AdS/CFT transfer matrix (5.5) in the $su(2)$ grading are given by (5.7), which we now abbreviate as follows

$$\tilde{\Lambda}_{su2}(\lambda)=\prod_{i=1}^{N} S_0(\lambda,p_i)^2\left[\Lambda_1(\lambda)-\Lambda_2(\lambda)-\Lambda_3(\lambda)+\Lambda_4(\lambda)\right]\left[\dot{\Lambda}_1(\lambda)-\dot{\Lambda}_2(\lambda)-\dot{\Lambda}_3(\lambda)+\dot{\Lambda}_4(\lambda)\right] \tag{E.1}$$

In order to obtain the corresponding expression in the $sl(2)$ grading, we perform a dualization on the fermionic roots $x^+(\lambda_j)$ [20, 21, 54] by noting that the first Bethe equation in (5.9) is an algebraic equation $q(x^+(\lambda_j))=0$, where $q(x)$ is given by

$$\begin{aligned} q(x) &= c_1\prod_{i=1}^{N}\eta(p_i)(x-x^-(p_i))\prod_{l=1}^{m_2} x\left(x+\frac{1}{x}-\tilde{\mu}_l-\frac{i}{2g}\right)\\ &\quad-\prod_{i=1}^{N}(x-x^+(p_i))\prod_{l=1}^{m_2} x\left(x+\frac{1}{x}-\tilde{\mu}_l+\frac{i}{2g}\right). \end{aligned} \tag{E.2}$$

Note that we have included a factor x^{m_2} to ensure that $q(x)$ is a polynomial in x of degree $N+2m_2$. This polynomial has m_1 roots $x^+(\lambda_j)$ and $\tilde{m}_1$ additional roots $x^+(\tilde{\lambda}_j)$, where $\tilde{m}_1=N+2m_2-m_1$. It can therefore be written also in the following factorized form

$$q(x)=c\prod_{j=1}^{m_1}\left(x-x^+(\lambda_j)\right)\prod_{j=1}^{\tilde{m}_1}\left(x-x^+(\tilde{\lambda}_j)\right), \tag{E.3}$$

where c is some non-vanishing constant. The equality of (E.2) and (E.3) implies that the function $Q(x)$ defined by

$$\begin{aligned} Q(x)=\prod_{j=1}^{m_1}\frac{1}{(x-x^+(\lambda_j))}\prod_{j=1}^{\tilde{m}_1}\frac{1}{(x-x^+(\tilde{\lambda}_j))}&\left[c_1\prod_{i=1}^{N}\eta(p_i)(x-x^-(p_i))\prod_{l=1}^{m_2}x\left(x+\frac{1}{x}-\tilde{\mu}_l-\frac{i}{2g}\right)\right.\\ &\left.-\prod_{i=1}^{N}(x-x^+(p_i))\prod_{l=1}^{m_2}x\left(x+\frac{1}{x}-\tilde{\mu}_l+\frac{i}{2g}\right)\right] \end{aligned} \tag{E.4}$$

is independent of x. The identities following from $Q\left(x^+(\lambda)\right)=Q\left(x^-(\lambda)\right)$ and $Q\left(1/x^+(\lambda)\right)=Q\left(1/x^-(\lambda)\right)$ imply that

$$\begin{aligned} \Lambda_1(\lambda)-\Lambda_2(\lambda) &= \eta(\lambda)^{m_1-N}\prod_{i=1}^{N}\frac{x^+(\lambda)-x^-(p_i)}{x^-(\lambda)-x^+(p_i)}\prod_{j=1}^{m_1}\frac{x^-(\lambda)-x^+(\lambda_j)}{x^+(\lambda)-x^+(\lambda_j)}\\ &\quad\times\left[c_1\prod_{i=1}^{N}\eta(p_i)-\prod_{i=1}^{N}\frac{x^+(\lambda)-x^+(p_i)}{x^+(\lambda)-x^-(p_i)}\prod_{l=1}^{m_2}\frac{x^+(\lambda)+\frac{1}{x^+(\lambda)}-\tilde{\mu}_l+\frac{i}{2g}}{x^+(\lambda)+\frac{1}{x^+(\lambda)}-\tilde{\mu}_l-\frac{i}{2g}}\right]\\ &= -\eta(\lambda)^{2m_2}\left[\frac{x^-(\lambda)}{x^+(\lambda)}\right]^{m_2}\prod_{j=1}^{\tilde{m}_1}\frac{1}{\eta(\lambda)}\frac{x^+(\lambda)-x^+(\tilde{\lambda}_j)}{x^-(\lambda)-x^+(\tilde{\lambda}_j)}\\ &\quad\times\left[1-c_1\prod_{i=1}^{N}\eta(p_i)\frac{x^-(\lambda)-x^-(p_i)}{x^-(\lambda)-x^+(p_i)}\prod_{l=1}^{m_2}\frac{x^-(\lambda)+\frac{1}{x^-(\lambda)}-\tilde{\mu}_l-\frac{i}{2g}}{x^-(\lambda)+\frac{1}{x^-(\lambda)}-\tilde{\mu}_l+\frac{i}{2g}}\right]\\ &\equiv \tilde{\Lambda}_1(\lambda)-\tilde{\Lambda}_2(\lambda)\,, \end{aligned} \tag{E.5}$$

and

$$
\begin{aligned}
-\Lambda_3(\lambda)+\Lambda_4(\lambda) &= \eta(\lambda)^{m_1-N}\prod_{i=1}^{N}\frac{x^+(\lambda)-x^+(p_i)}{x^-(\lambda)-x^+(p_i)}\prod_{j=1}^{m_1}\frac{\frac{1}{x^+(\lambda)}-x^+(\lambda_j)}{\frac{1}{x^-(\lambda)}-x^+(\lambda_j)} \\
&\quad\times\left[-\prod_{l=1}^{m_2}\frac{x^-(\lambda)+\frac{1}{x^-(\lambda)}-\tilde\mu_l-\frac{i}{2g}}{x^-(\lambda)+\frac{1}{x^-(\lambda)}-\tilde\mu_l+\frac{i}{2g}}+c_1^{-1}\prod_{i=1}^{N}\frac{1-\frac{1}{x^-(\lambda)x^+(p_i)}}{1-\frac{1}{x^-(\lambda)x^-(p_i)}}\eta(p_i)\right] \\
&= -\eta(\lambda)^{2m_2}\left[\frac{x^-(\lambda)}{x^+(\lambda)}\right]^{m_2}\prod_{j=1}^{\tilde m_1}\frac{1}{\eta(\lambda)}\frac{\frac{1}{x^-(\lambda)}-x^+(\tilde\lambda_j)}{\frac{1}{x^+(\lambda)}-x^+(\tilde\lambda_j)}\prod_{i=1}^{N}\frac{x^-(\lambda)-x^-(p_i)}{x^-(\lambda)-x^+(p_i)} \\
&\quad\times\left[-c_1^{-1}\prod_{i=1}^{N}\eta(p_i)\prod_{l=1}^{m_2}\frac{x^+(\lambda)+\frac{1}{x^+(\lambda)}-\tilde\mu_l+\frac{i}{2g}}{x^+(\lambda)+\frac{1}{x^+(\lambda)}-\tilde\mu_l-\frac{i}{2g}}+\prod_{i=1}^{N}\frac{1-\frac{1}{x^+(\lambda)x^-(p_i)}}{1-\frac{1}{x^+(\lambda)x^+(p_i)}}\right] \\
&\equiv -\tilde\Lambda_3(\lambda)+\tilde\Lambda_4(\lambda)\,,
\end{aligned}
\tag{E.6}
$$

respectively. Performing an analogous dualization for the roots $x^+(\dot\lambda_j)$ in $\dot\Lambda_1(\lambda)-\dot\Lambda_2(\lambda)$ and $-\dot\Lambda_3(\lambda)+\dot\Lambda_4(\lambda)$, and recalling that $\eta(\lambda)=\sqrt{\frac{x^+(\lambda)}{x^-(\lambda)}}$, we arrive at the desired result for the dual eigenvalues of the twisted AdS/CFT transfer matrix

$$
\begin{aligned}
\tilde\Lambda_{sl2}(\lambda) &= \prod_{i=1}^{N}S_0(\lambda,p_i)^2\left[\prod_{j=1}^{\tilde m_1}\frac{1}{\eta(\lambda)}\frac{x^+(\lambda)-x^+(\tilde\lambda_j)}{x^-(\lambda)-x^+(\tilde\lambda_j)}-\prod_{i=1}^{N}\frac{x^-(\lambda)-x^-(p_i)}{x^-(\lambda)-x^+(p_i)}\eta(p_i)\right. \\
&\quad\times\left\{c_1\prod_{j=1}^{\tilde m_1}\frac{1}{\eta(\lambda)}\left[\frac{x^+(\lambda)-x^+(\tilde\lambda_j)}{x^-(\lambda)-x^+(\tilde\lambda_j)}\right]\prod_{l=1}^{m_2}\frac{x^-(\lambda)+\frac{1}{x^-(\lambda)}-\tilde\mu_l-\frac{i}{2g}}{x^-(\lambda)+\frac{1}{x^-(\lambda)}-\tilde\mu_l+\frac{i}{2g}}\right. \\
&\quad\left.+c_1^{-1}\prod_{j=1}^{\tilde m_1}\frac{1}{\eta(\lambda)}\left[\frac{x^+(\tilde\lambda_j)-\frac{1}{x^-(\lambda)}}{x^+(\tilde\lambda_j)-\frac{1}{x^+(\lambda)}}\right]\prod_{l=1}^{m_2}\frac{x^+(\lambda)+\frac{1}{x^+(\lambda)}-\tilde\mu_l+\frac{i}{2g}}{x^+(\lambda)+\frac{1}{x^+(\lambda)}-\tilde\mu_l-\frac{i}{2g}}\right\} \\
&\quad\left.+\prod_{i=1}^{N}\left[\frac{1-\frac{1}{x^+(\lambda)x^-(p_i)}}{1-\frac{1}{x^+(\lambda)x^+(p_i)}}\right]\left[\frac{x^-(p_i)-x^-(\lambda)}{x^+(p_i)-x^-(\lambda)}\right]\prod_{j=1}^{\tilde m_1}\frac{1}{\eta(\lambda)}\left[\frac{x^+(\tilde\lambda_j)-\frac{1}{x^-(\lambda)}}{x^+(\tilde\lambda_j)-\frac{1}{x^+(\lambda)}}\right]\right] \\
&\quad\times\left[\prod_{j=1}^{\dot{\tilde m}_1}\frac{1}{\eta(\lambda)}\frac{x^+(\lambda)-x^+(\dot{\tilde\lambda}_j)}{x^-(\lambda)-x^+(\dot{\tilde\lambda}_j)}-\prod_{i=1}^{N}\frac{x^-(\lambda)-x^-(p_i)}{x^-(\lambda)-x^+(p_i)}\eta(p_i)\right. \\
&\quad\times\left\{c_2\prod_{j=1}^{\dot{\tilde m}_1}\frac{1}{\eta(\lambda)}\left[\frac{x^+(\lambda)-x^+(\dot{\tilde\lambda}_j)}{x^-(\lambda)-x^+(\dot{\tilde\lambda}_j)}\right]\prod_{l=1}^{\dot m_2}\frac{x^-(\lambda)+\frac{1}{x^-(\lambda)}-\dot{\tilde\mu}_l-\frac{i}{2g}}{x^-(\lambda)+\frac{1}{x^-(\lambda)}-\dot{\tilde\mu}_l+\frac{i}{2g}}\right. \\
&\quad\left.+c_2^{-1}\prod_{j=1}^{\dot{\tilde m}_1}\frac{1}{\eta(\lambda)}\left[\frac{x^+(\dot{\tilde\lambda}_j)-\frac{1}{x^-(\lambda)}}{x^+(\dot{\tilde\lambda}_j)-\frac{1}{x^+(\lambda)}}\right]\prod_{l=1}^{\dot m_2}\frac{x^+(\lambda)+\frac{1}{x^+(\lambda)}-\dot{\tilde\mu}_l+\frac{i}{2g}}{x^+(\lambda)+\frac{1}{x^+(\lambda)}-\dot{\tilde\mu}_l-\frac{i}{2g}}\right\} \\
&\quad\left.+\prod_{i=1}^{N}\left[\frac{1-\frac{1}{x^+(\lambda)x^-(p_i)}}{1-\frac{1}{x^+(\lambda)x^+(p_i)}}\right]\left[\frac{x^-(p_i)-x^-(\lambda)}{x^+(p_i)-x^-(\lambda)}\right]\prod_{j=1}^{\dot{\tilde m}_1}\frac{1}{\eta(\lambda)}\left[\frac{x^+(\dot{\tilde\lambda}_j)-\frac{1}{x^-(\lambda)}}{x^+(\dot{\tilde\lambda}_j)-\frac{1}{x^+(\lambda)}}\right]\right],
\end{aligned}
\tag{E.7}
$$

where the twist factors can be expressed in terms of $\tilde m_1$ and $\dot{\tilde m}_1$ as follows

$$
c_1=e^{i(\gamma_3-\gamma_2)J/2}e^{i\gamma_1(\dot{\tilde m}_1-2\dot m_2)/2}\,,\qquad c_2=e^{i(\gamma_3+\gamma_2)J/2}e^{-i\gamma_1(\tilde m_1-2m_2)/2}\,. \tag{E.8}
$$

If we identify c_1 (c_2) with $e^{i\alpha_l}$ ($e^{i\alpha_r}$), the eigenvalue of the "left (right) wing" transfer matrix matches the expression (8.1) of [17] with $Q = 1$, once we map $x^{\pm} \to x^{\mp}$, $g \to -g/2$, $x^+(\tilde{\lambda}_j) \to y_j^1$ ($x^+(\dot{\lambda}_j) \to y_j^2$), $\tilde{\mu}_l \to w_l^1$ ($\dot{\mu}_l \to w_l^2$), and $N = K^{\mathrm{I}}$, $\tilde{m}_1 = K_1^{\mathrm{II}}$, $m_2 = K_1^{\mathrm{III}}$ ($\dot{m}_1 = K_2^{\mathrm{II}}$, $\dot{m}_2 = K_2^{\mathrm{III}}$):

$$
\begin{aligned}
T_{1,1}^{l}(v\,|\,\vec{u}) = & \prod_{i=1}^{K_1^{\mathrm{II}}} \frac{y_i^1 - x^-}{y_i^1 - x^+} \sqrt{\frac{x^+}{x^-}} + \\
& + \prod_{i=1}^{K_1^{\mathrm{II}}} \frac{y_i^1 - x^-}{y_i^1 - x^+} \sqrt{\frac{x^+}{x^-}} \left[\frac{x^+ + \frac{1}{x^+} - y_i^1 - \frac{1}{y_i^1}}{x^+ + \frac{1}{x^+} - y_i^1 - \frac{1}{y_i^1} - \frac{2i}{g}} \right] \prod_{i=1}^{K^{\mathrm{I}}} \left[\frac{(x^- - x_i^-)(1 - x^- x_i^+)}{(x^+ - x_i^-)(1 - x^+ x_i^+)} \frac{x^+}{x^-} \right] \\
& - \prod_{i=1}^{K_1^{\mathrm{II}}} \frac{y_i^1 - x^-}{y_i^1 - x^+} \sqrt{\frac{x^+}{x^-}} \prod_{i=1}^{K^{\mathrm{I}}} \frac{x^+ - x_i^+}{x^+ - x_i^-} \sqrt{\frac{x_i^-}{x_i^+}} \times \\
& \times \left\{ e^{i\alpha_l} \prod_{i=1}^{K_1^{\mathrm{III}}} \frac{w_i^1 - x^+ - \frac{1}{x^+} - \frac{i}{g}}{w_i^1 - x^+ - \frac{1}{x^+} + \frac{i}{g}} + e^{-i\alpha_l} \prod_{i=1}^{K_1^{\mathrm{II}}} \frac{y_i^1 + \frac{1}{y_i^1} - x^+ - \frac{1}{x^+}}{y_i^1 + \frac{1}{y_i^1} - x^+ - \frac{1}{x^+} + \frac{2i}{g}} \prod_{i=1}^{K_1^{\mathrm{III}}} \frac{w_i^! - x^+ - \frac{1}{x^+} + \frac{3i}{g}}{w_i^1 - x^+ - \frac{1}{x^+} + \frac{i}{g}} \right\},
\end{aligned}
\tag{E.9}
$$

where we use the identities:

$$
\begin{aligned}
& \frac{y_i - x^-}{y_i - x^+} \left[\frac{x^+ + \frac{1}{x^+} - y_i - \frac{1}{y_i}}{x^+ + \frac{1}{x^+} - y_i - \frac{1}{y_i} - \frac{2i}{g}} \right] = \frac{y_i - \frac{1}{x^+}}{y_i - \frac{1}{x^-}}, \quad \frac{(x^- - x_i^-)(1 - x^- x_i^+)}{(x^+ - x_i^-)(1 - x^+ x_i^+)} \frac{x^+}{x^-} = \frac{(x^+ - x_i^+)\left(1 - \frac{1}{x^- x_i^+}\right)}{(x^+ - x_i^-)\left(1 - \frac{1}{x^- x_i^-}\right)}, \\
& \qquad \frac{w_i - x^+ - \frac{1}{x^+} + \frac{3i}{g}}{w_i - x^+ - \frac{1}{x^+} + \frac{i}{g}} = \frac{w_i - x^- - \frac{1}{x^-} + \frac{i}{g}}{w_i - x^- - \frac{1}{x^-} - \frac{i}{g}}.
\end{aligned}
\tag{E.10}
$$

The Bethe equations corresponding to the eigenvalues (E.7) are given by:

$$
\begin{aligned}
& c_1 \prod_{i=1}^{N} \left[\frac{x^+(\tilde{\lambda}_j) - x^-(p_i)}{x^+(\tilde{\lambda}_j) - x^+(p_i)} \right] \eta(p_i) = \prod_{l=1}^{m_2} \frac{x^+(\tilde{\lambda}_j) + \frac{1}{x^+(\tilde{\lambda}_j)} - \tilde{\mu}_l + \frac{i}{2g}}{x^+(\tilde{\lambda}_j) + \frac{1}{x^+(\tilde{\lambda}_j)} - \tilde{\mu}_l - \frac{i}{2g}}, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, \ldots, \tilde{m}_1, \\
& c_2 \prod_{i=1}^{N} \left[\frac{x^+(\dot{\lambda}_j) - x^-(p_i)}{x^+(\dot{\lambda}_j) - x^+(p_i)} \right] \eta(p_i) = \prod_{l=1}^{\dot{m}_2} \frac{x^+(\dot{\lambda}_j) + \frac{1}{x^+(\dot{\lambda}_j)} - \dot{\mu}_l + \frac{i}{2g}}{x^+(\dot{\lambda}_j) + \frac{1}{x^+(\dot{\lambda}_j)} - \dot{\mu}_l - \frac{i}{2g}}, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, \ldots, \dot{m}_1, \\
& c_1^2 \prod_{j=1}^{\tilde{m}_1} \frac{\tilde{\mu}_l - x^+(\tilde{\lambda}_j) - \frac{1}{x^+(\tilde{\lambda}_j)} + \frac{i}{2g}}{\tilde{\mu}_l - x^+(\tilde{\lambda}_j) - \frac{1}{x^+(\tilde{\lambda}_j)} - \frac{i}{2g}} = \prod_{\substack{k=1 \\ k \neq l}}^{m_2} \frac{\tilde{\mu}_l - \tilde{\mu}_k + \frac{i}{g}}{\tilde{\mu}_l - \tilde{\mu}_k - \frac{i}{g}}, \quad l = 1, \ldots, m_2, \\
& c_2^2 \prod_{j=1}^{\dot{m}_1} \frac{\dot{\mu}_l - x^+(\dot{\lambda}_j) - \frac{1}{x^+(\dot{\lambda}_j)} + \frac{i}{2g}}{\dot{\mu}_l - x^+(\dot{\lambda}_j) - \frac{1}{x^+(\dot{\lambda}_j)} - \frac{i}{2g}} = \prod_{\substack{k=1 \\ k \neq l}}^{m_2} \frac{\dot{\mu}_l - \dot{\mu}_k + \frac{i}{g}}{\dot{\mu}_l - \dot{\mu}_k - \frac{i}{g}}, \quad l = 1, \ldots, \dot{m}_2.
\end{aligned}
\tag{E.11}
$$

which match the equations (8.2) in [17], via the identifications used above. Following the change in notation in MM (47)-(49), (51), so that $\tilde{m}_1 \mapsto \tilde{m}_1^{(1)}, m_2 \mapsto m_2^{(1)}, \dot{m}_1 \mapsto$

$\tilde{m}_1^{(2)}, \dot{m}_2 \mapsto m_2^{(2)}$ and

$$N = K_4\,, \quad \tilde{m}_1^{(1)} = \tilde{K}_1 + \tilde{K}_3\,, \quad \tilde{m}_1^{(2)} = \tilde{K}_5 + \tilde{K}_7\,, \quad m_2^{(1)} = K_2\,, \quad m_2^{(2)} = K_6\,, \tag{E.12}$$

then the coefficients (E.8) are given by

$$c_1 = e^{i(\gamma_3-\gamma_2)J/2} e^{i\gamma_1(\tilde{K}_5+\tilde{K}_7-2K_6)/2}\,, \qquad c_2 = e^{i(\gamma_3+\gamma_2)J/2} e^{-i\gamma_1(\tilde{K}_1+\tilde{K}_3-2K_2)/2}\,, \tag{E.13}$$

and the first two equations in (E.11) become

$$c_1 e^{-iP/2} \prod_{i=1}^{K_4} \frac{1 - \frac{g^2}{x_{4,i}^- x_{\tilde{1},j}}}{1 - \frac{g^2}{x_{4,i}^+ x_{\tilde{1},j}}} \prod_{l=1}^{K_2} \frac{u_{\tilde{1},j} - u_{2,l} - \frac{i}{2}}{u_{\tilde{1},j} - u_{2,l} + \frac{i}{2}} = 1, \quad j = 1, \ldots, \tilde{K}_1 \tag{E.14}$$

$$c_1 e^{iP/2} \prod_{i=1}^{K_4} \frac{x_{4,i}^- - x_{\tilde{3},j}}{x_{4,i}^+ - x_{\tilde{3},j}} \prod_{l=1}^{K_2} \frac{u_{\tilde{3},j} - u_{2,l} - \frac{i}{2}}{u_{\tilde{3},j} - u_{2,l} + \frac{i}{2}} = 1, \quad j = 1, \ldots, \tilde{K}_3 \tag{E.15}$$

$$c_2 e^{iP/2} \prod_{i=1}^{K_4} \frac{x_{4,i}^- - x_{\tilde{5},j}}{x_{4,i}^+ - x_{\tilde{5},j}} \prod_{l=1}^{K_6} \frac{u_{\tilde{5},j} - u_{6,l} - \frac{i}{2}}{u_{\tilde{5},j} - u_{6,l} + \frac{i}{2}} = 1, \quad j = 1, \ldots, \tilde{K}_5 \tag{E.16}$$

$$c_2 e^{-iP/2} \prod_{i=1}^{K_4} \frac{1 - \frac{g^2}{x_{4,i}^- x_{\tilde{7},j}}}{1 - \frac{g^2}{x_{4,i}^+ x_{\tilde{7},j}}} \prod_{l=1}^{K_6} \frac{u_{\tilde{7},j} - u_{6,l} - \frac{i}{2}}{u_{\tilde{7},j} - u_{6,l} + \frac{i}{2}} = 1, \quad j = 1, \ldots, \tilde{K}_7 \tag{E.17}$$

while the last two can be written as

$$c_1^{-2} \prod_{j=1}^{\tilde{K}_1} \frac{u_{2,l} - u_{\tilde{1},j} - \frac{i}{2}}{u_{2,l} - u_{\tilde{1},j} + \frac{i}{2}} \prod_{j=1}^{\tilde{K}_3} \frac{u_{2,l} - u_{\tilde{3},j} - \frac{i}{2}}{u_{2,l} - u_{\tilde{3},j} + \frac{i}{2}} \prod_{\substack{k=1\\k\neq l}}^{K_2} \frac{u_{2,l} - u_{2,k} + i}{u_{2,l} - u_{2,k} - i} = 1, \quad l = 1, \ldots, K_2 \tag{E.18}$$

$$c_2^{-2} \prod_{j=1}^{\tilde{K}_5} \frac{u_{6,l} - u_{\tilde{5},j} - \frac{i}{2}}{u_{6,l} - u_{\tilde{5},j} + \frac{i}{2}} \prod_{j=1}^{\tilde{K}_7} \frac{u_{6,l} - u_{\tilde{7},j} - \frac{i}{2}}{u_{6,l} - u_{\tilde{7},j} + \frac{i}{2}} \prod_{\substack{k=1\\k\neq l}}^{K_6} \frac{u_{6,l} - u_{6,k} + i}{u_{6,l} - u_{6,k} - i} = 1, \quad l = 1, \ldots, K_6 \tag{E.19}$$

Moreover, the equations for the type-4 Bethe roots turn out to be undeformed:

$$\begin{aligned} e^{-ip_k L} &= \tilde{\Lambda}_{sl2}(p_k) \\ &= \prod_{i=1}^{N} S_0^2(p_k, p_i) \prod_{j=1}^{\tilde{m}_1} \frac{1}{\eta(p_k)} \frac{x^+(p_k) - x^+(\tilde{\lambda}_j)}{x^-(p_k) - x^+(\tilde{\lambda}_j)} \prod_{j=1}^{\dot{m}_1} \frac{1}{\eta(p_k)} \frac{x^+(p_k) - x^+(\dot{\tilde{\lambda}}_j)}{x^-(p_k) - x^+(\dot{\tilde{\lambda}}_j)}\,. \end{aligned} \tag{E.20}$$

One can recover the equations (E.11) and (E.20) also by dualizing directly (using the relations (E.5), (E.6)) the corresponding Bethe equations in the $su(2)$ grading, namely (5.9) and (5.20). Setting again $L = -J$, substituting the result for the scalar factor in MM (36) and changing notations, we obtain

$$
\begin{aligned}
e^{ip_k\left[J+\frac{1}{2}(\tilde{K}_3-\tilde{K}_1)+\frac{1}{2}(\tilde{K}_5-\tilde{K}_7)\right]} = & \prod_{\substack{i=1\\ i\neq k}}^{K_4}\left[\frac{x^-_{4,k}-x^+_{4,i}}{x^+_{4,k}-x^-_{4,i}}\right]\left[\frac{1-\frac{g^2}{x^+_{4,k}x^-_{4,i}}}{1-\frac{g^2}{x^-_{4,k}x^+_{4,i}}}\right][\sigma(p_k,p_i)]^2 \\
& \times\prod_{j=1}^{\tilde{K}_3}\frac{x^+_{4,k}-x_{\tilde{3},j}}{x^-_{4,k}-x_{\tilde{3},j}}\prod_{j=1}^{\tilde{K}_1}\frac{1-\frac{g^2}{x^+_{4,k}x_{\tilde{1},j}}}{1-\frac{g^2}{x^-_{4,k}x_{\tilde{1},j}}} \\
& \times\prod_{j=1}^{\tilde{K}_5}\frac{x^+_{4,k}-x_{\tilde{5},j}}{x^-_{4,k}-x_{\tilde{5},j}}\prod_{j=1}^{\tilde{K}_7}\frac{1-\frac{g^2}{x^+_{4,k}x_{\tilde{7},j}}}{1-\frac{g^2}{x^-_{4,k}x_{\tilde{7},j}}}, \quad k=1,\ldots,K_4\,. \qquad \text{(E.21)}
\end{aligned}
$$

We can now recover the corresponding untwisted Bethe equations of Beisert and Staudacher [18] in the grading $\eta_1=\eta_2=-1$ by setting $P=0$ and recalling (see (5.6) in [18]) the definition of the angular momentum charge for that grading

$$
J=\mathcal{L}-\frac{1}{2}(\tilde{K}_3-\tilde{K}_1)-\frac{1}{2}(\tilde{K}_5-\tilde{K}_7)\,. \qquad \text{(E.22)}
$$

E.1 Comparison with BR

Since BR [9] does not explicitly consider the all-loop twisted Bethe equations in the sl_2 grading, a little more effort is required to make the comparison. The BR Bethe equations in this grading are still given by (6.2), where now

$$
U_{\tilde{1}}(x)=U^{-1}_{\tilde{3}}(x)=U^{-1}_{\tilde{5}}(x)=U_{\tilde{7}}(x)=\prod_{k=1}^{K_4}S^{-1}_{\text{aux}}(x_{4,k},x) \qquad \text{(E.23)}
$$

and

$$
\begin{aligned}
U_4(x) = U_{\text{s}}(x)\left(\frac{x^-}{x^+}\right)^{\mathcal{L}}\prod_{k=1}^{K_4}S^2_{\text{aux}}(x,x_{4,k})\prod_{k=1}^{\tilde{K}_1}S_{\text{aux}}(x,x_{\tilde{1},k})\prod_{k=1}^{\tilde{K}_3}S^{-1}_{\text{aux}}(x,x_{\tilde{3},k}) \\
\times\prod_{k=1}^{\tilde{K}_5}S^{-1}_{\text{aux}}(x,x_{\tilde{5},k})\prod_{k=1}^{\tilde{K}_7}S_{\text{aux}}(x,x_{\tilde{7},k}). \qquad \text{(E.24)}
\end{aligned}
$$

(The quantities $U_0\,, S_{\text{aux}}\,, U_{\text{s}}$ are the same as before, see (6.3), (6.5).)

For the sl_2 grading with $\eta_1=\eta_2=-1$ which we now consider, $M_{j,j'}$ is the Cartan matrix specified by figure 2 (see eq. (5.1) in [18]). The twist matrix $\mathbf{A}$ is given by BR (5.19)[14]

$$
\mathbf{A}=\delta_1\left(\mathbf{q}_p\mathbf{q}^{\mathrm{T}}_{q_2}-\mathbf{q}_{q_2}\mathbf{q}^{\mathrm{T}}_p\right)+\delta_2\left(\mathbf{q}_{q_2}\mathbf{q}^{\mathrm{T}}_{q_1}-\mathbf{q}_{q_1}\mathbf{q}^{\mathrm{T}}_{q_2}\right)+\delta_3\left(\mathbf{q}_{q_1}\mathbf{q}^{\mathrm{T}}_p-\mathbf{q}_p\mathbf{q}^{\mathrm{T}}_{q_1}\right)\,, \qquad \text{(E.25)}
$$

[14] We note that eqs. (6.2) for $\eta_1=\eta_2=-1$ can be alternatively obtained by writing eqs. (6.1) in [55] with $\eta=-1$, setting their twists as

$$
e^{i(\phi_i-\phi_{i+1})}=e^{-i(\mathbf{A}\mathbf{K})_i}\,,
$$

and, as suggested there, exchanging the twists $\phi_1\leftrightarrow\phi_2$, $\phi_3\leftrightarrow\phi_4$, $\phi_5\leftrightarrow\phi_6$, $\phi_7\leftrightarrow\phi_8$.

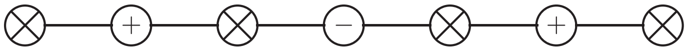

Figure 2. Dynkin diagram of $su(2,2|4)$ for $\eta_1 = \eta_2 = -1$.

where we take the three charge vectors to be

$$
\begin{aligned}
\mathbf{q}_{q_1} &= (\ \ 0|+1,-2,+1,\ \ 0,\ \ 0,\ \ 0,\ \ 0),\\
\mathbf{q}_{p} &= (+1|\ \ 0,+1,-1,\ \ 0,-1,+1,\ \ 0),\\
\mathbf{q}_{q_2} &= (\ \ 0|\ \ 0,\ \ 0,\ \ 0,\ \ 0,+1,-2,+1),
\end{aligned}
\tag{E.26}
$$

such that the Dynkin labels $[q_1, p, q_2]$ in the grading $\eta_1 = \eta_2 = -1$ (see, for instance, eq. (5.3) in [18]) can be extracted as

$$
\mathbf{q}_{q_1} \cdot \mathbf{K} = q_1, \qquad \mathbf{q}_{p} \cdot \mathbf{K} = p, \qquad \mathbf{q}_{q_2} \cdot \mathbf{K} = q_2\,, \tag{E.27}
$$

where now $\mathbf{K} = (L|\,\tilde{K}_1, K_2, \tilde{K}_3, K_4, \tilde{K}_5, K_6, \tilde{K}_7)$. Explicitly, the twisting matrix $\mathbf{A}$ reads

$$
\mathbf{A} = \left(\begin{array}{c|cccccccc}
0 & -\delta_3 & +2\delta_3 & -\delta_3 & 0 & +\delta_1 & -2\delta_1 & +\delta_1 \\
\hline
+\delta_3 & 0 & +\delta_3 & -\delta_3 & 0 & -\delta_2-\delta_3 & +2\delta_2+\delta_3 & -\delta_2 \\
-2\delta_3 & -\delta_3 & 0 & +\delta_3 & 0 & +\delta_1+2\delta_2+2\delta_3 & -2\delta_1-4\delta_2-2\delta_3 & +\delta_1+2\delta_2 \\
+\delta_3 & +\delta_3 & -\delta_3 & 0 & 0 & -\delta_1-\delta_2-\delta_3 & +2\delta_1+2\delta_2+\delta_3 & -\delta_1-\delta_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\delta_1 & +\delta_2+\delta_3 & -\delta_1-2\delta_2-2\delta_3 & +\delta_1+\delta_2+\delta_3 & 0 & 0 & +\delta_1 & -\delta_1 \\
+2\delta_1 & -2\delta_2-\delta_3 & +2\delta_1+4\delta_2+2\delta_3 & -2\delta_1-2\delta_2-\delta_3 & 0 & -\delta_1 & 0 & +\delta_1 \\
-\delta_1 & +\delta_2 & -\delta_1-2\delta_2 & +\delta_1+\delta_2 & 0 & +\delta_1 & -\delta_1 & 0
\end{array}\right).
$$

It then follows from (6.6) and BR (4.27) that

$$
\begin{aligned}
(\mathbf{AK})_0 &= \frac{1}{2}\Big[\gamma_2\left(\tilde{K}_1+\tilde{K}_3-\tilde{K}_5-\tilde{K}_7-2K_2+2K_6\right)\\
&\qquad +\gamma_3\left(-\tilde{K}_1-\tilde{K}_3-\tilde{K}_5-\tilde{K}_7+2K_2+2K_6\right)\Big]\,,
\end{aligned}
\tag{E.28}
$$

$$
(\mathbf{AK})_1 - \frac{1}{2}(\mathbf{AK})_0 = \frac{1}{2}\left[(\gamma_3-\gamma_2)J + \gamma_1\left(\tilde{K}_5+\tilde{K}_7-2K_6\right)\right]\,, \tag{E.29}
$$

$$
(\mathbf{AK})_2 = -2\left[(\mathbf{AK})_1 - \frac{1}{2}(\mathbf{AK})_0\right]\,, \tag{E.30}
$$

$$
(\mathbf{AK})_3 + \frac{1}{2}(\mathbf{AK})_0 = (\mathbf{AK})_1 - \frac{1}{2}(\mathbf{AK})_0\,, \tag{E.31}
$$

$$
(\mathbf{AK})_4 = 0\,. \tag{E.32}
$$

$$
(\mathbf{AK})_5 + \frac{1}{2}(\mathbf{AK})_0 = \frac{1}{2}\left[(\gamma_3+\gamma_2)J - \gamma_1\left(\tilde{K}_1+\tilde{K}_3-2K_2\right)\right]\,, \tag{E.33}
$$

$$
(\mathbf{AK})_6 = -2\left[(\mathbf{AK})_5 + \frac{1}{2}(\mathbf{AK})_0\right]\,, \tag{E.34}
$$

$$
(\mathbf{AK})_7 - \frac{1}{2}(\mathbf{AK})_0 = (\mathbf{AK})_5 + \frac{1}{2}(\mathbf{AK})_0\,. \tag{E.35}
$$

As already noted in (6.14), the total momentum is given by $P = -(\mathbf{AK})_0$.

We now compare the BR Bethe equations with the ones which we derived above by dualization. The fact that eqs. (E.21) are not deformed matches with (E.32) and (6.2)

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with $j = 4$. Substituting for P using (6.14), and noting the following identities (proved using (E.8), (E.22) and (E.28)–(E.35)),

$$c_1 e^{-iP/2} = e^{i(\mathbf{AK})_1}\,, \quad c_1 e^{iP/2} = e^{i(\mathbf{AK})_3}\,, \quad c_2 e^{iP/2} = e^{i(\mathbf{AK})_5}\,, \quad c_2 e^{-iP/2} = e^{i(\mathbf{AK})_7}\,,$$
$$c_1^{-2} = e^{i(\mathbf{AK})_2}\,, \qquad c_2^{-2} = e^{i(\mathbf{AK})_6}\,, \tag{E.36}$$

we see that eqs. (E.14)–(E.19) match with (6.2) with $j = 1, 3, 5, 7, 2, 6$ respectively.

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