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Exact world-sheet S-matrices for AdS/CFT

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Exact S-matrices have played essential roles in nonperturbative computations arising in AdS/CFT dualities. We review basic aspects of the world-sheet S-matrices for the AdS/CFT and their applications in an introductory level. We focus mainly on the exact S-matrix of the planar $\mathcal{N}=4$ SYM theory which consists of AdS_5/CFT_4 and its applications. S-matrices of β- and η-deformations of AdS_5/CFT_4 and AdS_4/CFT_3 and related recent developments are briefly commented.

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I. INTRODUCTION

AdS/CFT duality conjecture, proposed by J. Maldacena in 1997 [1] and refined in subsequent papers [2,3], is an equivalence between type IIB superstring theory on $AdS_5 \times S^5$ background geometry and $\mathcal{N}=4$ supersymmetric $SU(N)$ Yang-Mills gauge theory (SYM) defined on 4d space-time which is the boundary of AdS_5. The duality has been already glimpsed in the $1/N$ expansion where its order is identified with the genus of 2d surfaces where the Feynman diagrams are drawn [4]. The qualitative gauge/string correspondence also appeared in the flux tubes connecting quarks in the strong expansions [5–7]. But the AdS/CFT duality shows explicit and quantitiative relations between the two theories. The radius R of $AdS_5 \times S^5$ space and the inverse string tension α' are related by $R^2/\alpha' = \sqrt{\lambda}$ in terms of the 't Hooft coupling constant λ defined by $\lambda = N g_{\rm YM}^2$ [4]. Since three-strings vertices vanish in the planar ('t Hooft) limit $N \to \infty$, each string state can be characterized by a few conserved quantities such as energy and angular momentum. The $\mathcal{N}=4$ SYM theory, being a conformal field theory (CFT) [8–10], contains composite operators with single trace over the color index in the planar limit which correspond to various string states moving in the curved background. Among correlation functions on both sides of the duality, most studies have been focused on two-point functions which are completely determined by conformal dimensions and energies of corresponding string states. While the conformal dimensions are calculable in ordinary perturbative series for small values of λ, the energies of string states can be computed in the large λ limit as $1/\lambda$ expansions based on the classic string actions on the world-sheet. This mismatch was at the core of difficulty in proving the AdS/CFT duality, and needed a new non-perturbative approach based on the integrability which is valid for any value of λ.

The integrability was first discovered in perturbative gauge and string theories. One-loop dilatation operator for the single trace composite operators in the $SO(6)$ scalar sector was identified with a hamiltonian of the $SO(6)$-invariant integrable spin chain, from which the conformal dimensions could be obtained by the Bethe ansatz [11]. The integrability of the dilatation operators is maintained in two- and three-loops [12,13]. The world-sheet string action on $AdS_5 \times S^5$ can be constructed as a sigma model on a coset space [14], and shown to be classically integrable [15]. These results have lead to conjecture that the integrability is exact in all orders of λ.

In this short review, we focus on the non-perturbative world-sheet S-matrices as a result of integrability [16–18]. (These should be distinguished from the space-time scattering amplitudes.) Once constructed by symmetry, the factorized elastic S-matrix [20] can be applied to compute the conformal dimensions, hence the energies of string states for any value of coupling constant, by integrability tools. Also, we explain how this approach can be extended to several deformed theories and AdS_4/CFT_3.

We refer [21,22] for more comprehensive reviews which include technical details and other important results not covered here.

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II. $\mathcal{N}=4$ SYM AND ADS$_5$/CFT$_4$

The action of the $\mathcal{N}=4$ SYM is given by

$$S = \int d^4x \ \mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mathcal{D}_\mu\Phi_i\mathcal{D}^\mu\Phi^i - \frac{g_{\rm YM}^2}{4}[\Phi_i,\Phi_j][\Phi^i,\Phi^j] + \overline{\Psi}^a\sigma^\mu\mathcal{D}_\mu\Psi_a - \frac{ig_{\rm YM}}{2}\sigma_i^{ab}\Psi_a[\Phi^i,\Psi_b] - \frac{ig_{\rm YM}}{2}\sigma^i_{ab}\overline{\Psi}^a[\Phi_i,\overline{\Psi}^b]\right), \quad (1)$$

where σ^μ and σ^i are the gamma matrices in four and six dimensions, respectively. There are 6 scalar fields Φ_i, 4 Dirac fields Ψ_a, and 4 covariant derivatives $\mathcal{D}_\mu$, all in the adjoint representations of the color $SU(N)$. The trace is taken over the color index. This theory has coformal symmetry with vanishing β function whose total symmetry group becomes $PSU(2,2|4)$ including $SU(4)$ R-symmetry.

The first breakthrough was made when operator-mixing matrix among composite operators consisted of scalar fields of the SYM was identified with an integrable spin-chain Hamiltonian in the leading λ order. The conformal dimensions of the eigen-operators can be determined by Bethe ansatz method which diagonalizes the Hamiltonian. This interesting interplay between a gauge theory and an integrable model has been quickly generalized to a most generic composite operators of the SYM and to higher-order perturbative computation, and eventually to conjectures of "asymptotic" Bethe ansatz which are claimed to be valid up to all orders.

Single-trace (gauge-invariant) composite operators consisted of the elementary fields

$$\mathrm{Tr}\left[\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_L\right](x), \quad \mathcal{O}_i \in \{\mathcal{D}^n\Phi\,, \mathcal{D}^n\Psi\,, D^nF\}\,, \quad (2)$$

are local (all the elementary fields are at the same location x), gauge-invariant and super-multiplets of the $PSU(2,2|4)$ symmetry. (We will omit the space-time coordinate x unless it is necessary.) Composite operators are transformed to others under scale transformations. The dilation operator which governs the mixing is unknown except a few lowest orders in perturbation theory. Even for the perturbative dilatation operator, it is still challenge to diagonalize the matrix to find eigenvalues as anormalous dimension.

A special case is the 1/2-BPS operator $\mathrm{Tr}[Z^L]$ ($Z \equiv \Phi_5 + i\Phi_6$), which has no quantum correction and the anormalous dimension is strictly zero in all orders. Then, it is possible to represent this protected operator by a vacuum state

$$\mathrm{Tr}[\overbrace{ZZZ\cdots ZZZ}^{L}] \equiv |0\rangle, \quad (3)$$

and generic composite operators (2), obtained by replacing each Z in (3) with any of the elementary SYM fields ("impurities"), as excited states

$$|\overset{1}{\downarrow}{Z}\cdots Z\overset{x_1}{\downarrow}{\chi}\, Z\cdots Z\overset{x_2}{\downarrow}{\chi'}\, Z\cdots Z\overset{x_M}{\downarrow}{\chi''}\, Z\cdots \overset{L}{\downarrow}{Z}\rangle = A^\dagger_\chi(x_1)A^\dagger_{\chi'}(x_2)\cdots A^\dagger_{\chi''}(x_M)|0\rangle. \quad (4)$$

Here $A^\dagger_\chi$ is a Zamolodchikov-Faddeev operator of χ [19] acting on the vacuum state and the indices x_i's denote locations of impurities in the chain of elementary fields. Due to the cyclic property of the trace, the state (39) should be invariant under a uniform translation $x_k \to x_k+1$.

Integrability in a quantum field theory is possible when it has infinitely many conserved charges, a difficult condition to meet, and usually appears in one dimension. For the planar $\mathcal{N}=4$ SYM, the magnons move along the spin chain with the location on the chain as a 1d coordinate. Charges Q_k, $k=1,\ldots,N$, which are expressed as polynormials of momentum (*e.g.* the energy is a quadratic), should be conserved

$$Q_k(\mathrm{in}) = \sum_{i=1}^n (p_i)^k = Q_k(\mathrm{out}) = \sum_{i=1}^m (p'_i)^k \quad (5)$$

where $p_1,\ldots,p_n$, and $p'_1,\ldots,p'_m$ are momenta before and after scattering, respectively. For sufficiently large (infinite) N, these conservation laws lead to [20]

$$n=m, \qquad \{p'_1,\ldots,p'_n\} = \{p_1,\ldots,p_n\}. \quad (6)$$

The first equality means the number of particles are preserved while the second indicates that only momentum exchanges are allowed in the scattering process. This implies also that the multi-particle scattering amplitudes are factorized into products of $2\to 2$ S-matrices. Furthermore, since three-particle process should be independent of orders of $2\to 2$ scatterings, they should satisfy the Yang-Baxter equation [23–25]. Many 2-particle S-matrices have been found exactly by either directly from the Yang-Baxter equation or by symmetries imposed on the on-shell states along with additional axioms such as unitarity and crossing symmetry. These (2-particle) S-matrices are functions of two momenta and coupling constants of the underlying theories.

For the AdS/CFT system, 2-particle states, Fourier-transformed in the momentum space, can be obtained by two Zamolodchikov-Faddeev operators acting on the BPS vacuum:

$$|\chi_1(p_1)\chi_2(p_2)\rangle \equiv A^\dagger_{\chi_1}(p_1)A^\dagger_{\chi_2}(p_2)|0\rangle, \quad (7)$$

and the $2\to 2$ S-matrix is matrix elements between incoming and out-going states [19,20]

$$|\chi_1(p_1)\chi_2(p_2)\rangle_{\rm in} \equiv \sum_{\chi'_1\chi'_2} S_{\chi_1\chi_2}^{\chi'_1\chi'_2}(p_1,p_2)|\chi'_1(p_1)\chi'_2(p_2)\rangle_{\rm out}. \quad (8)$$

In the case of $\mathcal{N}=4$ SYM, there are 16 excitations (8 Ψ_a's, 4 $\mathcal{D}_\mu$'s, and 4 of the 6 Φ_i's), hence the 2-particle

S-matrix has the size of $16^2 \times 16^2$. A very important observation is that the excitation spectrum in the single trace composite operator in the planar $\mathcal{N}=4$ SYM can be written as

$$\chi, \chi', \chi'', \ldots \in \{\Phi_{a\dot{a}}\,, \Psi_{\dot{a}\alpha}\,, \bar{\Psi}_{a\dot{\alpha}}\,, D_{\alpha\dot{\alpha}}Z\}\,, \quad a, \dot{a} = 1,2, \quad \alpha, \dot{\alpha} = 3,4\,. \tag{9}$$

which belong to a fundamental representation of a centrally extended $su(2|2)_L \otimes su(2|2)_R$ [18]. The creation operators $A^\dagger_\chi$ can be written as

$$A^\dagger_{i\dot{i}}(p) = A^\dagger_i(p) \otimes A^\dagger_{\dot{i}}(p), \quad i = (a,\alpha) = 1,2,3,4, \quad \dot{i} = (\dot{a},\dot{\alpha}) = 1,2,3,4. \tag{10}$$

Therefore, the AdS/CFT S-matrix is factorized into a tensor product of two identical S-matrices, one acting on the index i and the other on $\dot{i}$, $\mathbb{S} = S \otimes \dot{S}$. Since they are identical, we can focus on S which is $4^2 \times 4^2$ matrix.

The S-matrix elements are defined by

$$A^\dagger_i(p_1)\, A^\dagger_j(p_2)|0\rangle_{\rm in} = \sum_{i',j'=1}^{4} S_{i\,j}^{i'j'}(p_1,p_2)\, A^\dagger_{j'}(p_2)\, A^\dagger_{i'}(p_1)|0\rangle_{\rm out}. \tag{11}$$

By acting symmetry generators of $su(2|2)$ on both sides of Eq.(11) and assuming they annihilate the vacuum states, one can derive a system of linear equations for the S-matrix elements, which determine the S-matrix up to an overall scalar funtion S_0. Out of 16×16 elements, following 36 matrix elements are nonzero [18,26]

$$\begin{aligned}
S_{11}^{11} &= S_{22}^{22} = A\,, \quad S_{33}^{33} = S_{44}^{44} = D\,,\\
S_{12}^{12} &= S_{21}^{21} = \frac{1}{2}(A-B)\,, \quad S_{12}^{21} = S_{21}^{12} = \frac{1}{2}(A+B)\,,\\
S_{34}^{34} &= S_{43}^{43} = \frac{1}{2}(D-E)\,, \quad S_{34}^{43} = S_{43}^{34} = \frac{1}{2}(D+E)\,,\\
S_{12}^{34} &= S_{21}^{43} = -S_{12}^{43} = -S_{21}^{34} = -\frac{1}{2}\,C\,,\\
S_{34}^{12} &= S_{43}^{21} = -S_{34}^{21} = -S_{43}^{12} = -\frac{1}{2}\,F\,,\\
S_{13}^{13} &= S_{23}^{23} = S_{14}^{14} = S_{24}^{24} = G\,,\\
S_{13}^{31} &= S_{23}^{32} = S_{14}^{41} = S_{24}^{42} = H\,,\\
S_{31}^{13} &= S_{32}^{23} = S_{41}^{14} = S_{42}^{24} = K\,,\\
S_{31}^{31} &= S_{32}^{32} = S_{41}^{41} = S_{42}^{42} = L\,,
\end{aligned} \tag{12}$$

with

$$\begin{aligned}
A &= S_0 \frac{x_2^- - x_1^+}{x_2^+ - x_1^-}\frac{\eta_1\eta_2}{\tilde{\eta}_1\tilde{\eta}_2}\,, \quad B = -S_0\left[\frac{x_2^- - x_1^+}{x_2^+ - x_1^-} + 2\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\right]\frac{\eta_1\eta_2}{\tilde{\eta}_1\tilde{\eta}_2}\,,\\
C &= S_0 \frac{2i x_1^- x_2^- (x_1^+ - x_2^+)\eta_1\eta_2}{x_1^+ x_2^+ (x_1^- - x_2^+)(1 - x_1^- x_2^-)}\,, \quad D = -S_0\,, \quad E = S_0\left[1 - 2\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- + x_2^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\right]\,,\\
F &= S_0 \frac{2i(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- - x_2^+)(1 - x_1^- x_2^-)\tilde{\eta}_1\tilde{\eta}_2}\,, \quad G = S_0\frac{(x_2^- - x_1^-)}{(x_2^+ - x_1^-)}\frac{\eta_1}{\tilde{\eta}_1}\,,\\
H &= S_0 \frac{(x_2^+ - x_2^-)}{(x_1^- - x_2^+)}\frac{\eta_1}{\tilde{\eta}_2}\,, \quad K = S_0\frac{(x_1^+ - x_1^-)}{(x_1^- - x_2^+)}\frac{\eta_2}{\tilde{\eta}_1}\,, \quad L = S_0\frac{(x_1^+ - x_2^+)}{(x_1^- - x_2^+)}\frac{\eta_2}{\tilde{\eta}_2}\,,
\end{aligned} \tag{13}$$

and $x_i^\pm = x^\pm(p_i)\,, \quad \eta_1 = \eta(p_1)e^{ip_2/2}\,, \quad \eta_2 = \eta(p_2)\,, \quad \tilde{\eta}_1 = \eta(p_1)\,, \quad \tilde{\eta}_2 = \eta(p_2)e^{ip_1/2}$ where

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}\,, \qquad \frac{x^+}{x^-} = e^{ip}\,, \quad \eta = e^{ip/4}\sqrt{i(x^- - x^+)}\,. \tag{14}$$

The energy of a particle with momentum p is given by

$$E = -ig\left(x^+ - \frac{1}{x^+} - x^- + \frac{1}{x^-}\right) = \sqrt{1 + 16g^2\sin^2\frac{p}{2}}\,. \tag{15}$$

This result for the energy matches with known limits for $g = \sqrt{\lambda}/(4\pi)$ [27]. We remark that the above S-matrix is in fact in the "string frame" [26]. Another version is defined in the "spin-chain frame," where the S-matrix is still (13) but with $\eta_1 = \tilde{\eta}_1 = \eta(p_1)\,, \quad \eta_2 = \tilde{\eta}_2 = \eta(p_2)$ [18]. An interesting observation is that this $su(2|2)$ S-matrix is closely related to Shastry's R-matrix [28] which establishes integrability for the Hubbard model [29].

This S-matrix can be shown to satisfy the Yang-Baxter equation (or "twisted" version of it for the spin-chain frame S-matrix) and the unitarity $S_{12}(p_1,p_2)\,S_{21}(p_2,p_1) = \mathbb{I}$ ($\mathbb{I}$ is the 16×16 identity matrix) if its scalar factor S_0 satisfies a functional relation

$$S_0(p_1,p_2)\,S_0(p_2,p_1) = 1\,. \tag{16}$$

Another condition for S_0 is the crossing symmetry [30–32], which can be understood as a commutativity of the particle creation operator $A^\dagger_i$ with 2-particle singlet operator [32] defined by

$$\begin{aligned}
\mathbb{I}(p) &\equiv C^{ij}A^\dagger_i(p)A^\dagger_j(\bar{p})\\
&= -i\epsilon^{ab}A^\dagger_a(p)A^\dagger_b(\bar{p}) + \epsilon^{\alpha\beta}A^\dagger_\alpha(p)A^\dagger_\beta(\bar{p}).
\end{aligned} \tag{17}$$

Here, $\bar{p}$ is defined by $x^{\pm}(\bar{p}) = 1/x^{\pm}(p)$ so that $E(\bar{p}) = -E$, $\bar{p} = -p$. Nontrivial condition $A_i^{\dagger}(p_1)\mathbb{I}(p_2) = \mathbb{I}(p_2)A_i^{\dagger}(p_1)$ can be satisfied if

$$S_0(p_1,p_2)S_0(p_1,\bar{p}_2) = \frac{x_1^- - x_2^+}{x_1^+ - x_2^+}\,\frac{\frac{1}{x_1^-} - x_2^-}{\frac{1}{x_1^+} - x_2^-}. \qquad (18)$$

Going through marvelous mathematical manipulations, unique solution (up to "CDD" ambiguity) of Eqs. (16) and (18) has been found

$$S_0(p_1,p_2)^2 = \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\,\frac{1 - \frac{1}{x_1^+ x_2^-}}{1 - \frac{1}{x_1^- x_2^+}}\,\sigma(p_1,p_2)^2, \qquad (19)$$

where the BES phase factor $\sigma(p_1,p_2)$ [33–36] is

$$\sigma(p_1,p_2) = \exp\left\{i[\chi(x_1^+,x_2^-) + \chi(x_1^-,x_2^+) - \chi(x_1^+,x_2^+) - \chi(x_1^-,x_2^-)]\right\}$$
$$\chi(x,y) \equiv -i\oint_{|z|=1}\frac{dz}{2\pi i}\oint_{|z|=1}\frac{dz'}{2\pi i}\,\frac{1}{x-z}\,\frac{1}{y-z'}\,\frac{\ln\Gamma\left[1 + ig\left(z + \frac{1}{z} - z' - \frac{1}{z'}\right)\right]}{\ln\Gamma\left[1 - ig\left(z + \frac{1}{z} - z' - \frac{1}{z'}\right)\right]}\,. \qquad (20)$$

This S-matrix can be checked perturbatively. In the strong coupling, the S-matrix on worldsheet can be computed by using light-cone gauge fixed Lagrangian [37]. The S-matrix has bound-state poles [35, 38–40] which belong to atypical totally symmetric representations of $su(2|2)$ with dimension $4Q$ [41] with a generalized dispersion relation

$$E = \sqrt{Q^2 + 16g^2\sin^2\frac{p}{2}}\,. \qquad (21)$$

The S-matrices of the bound states are not completely fixed by the $su(2|2)$ symmetry. An important observation is that the fundamental bulk S-matrix (12),(13) has a remarkable Yangian symmetry $Y(su(2|2))$ [42] which determines completely the two-particle [41,43] and general Q-particle bound state bulk S-matrices [44].

III. APPLICATIONS OF THE S-MATRIX

The world-sheet S-matrix is scattering amplitudes of excitations on the 2d world-sheet nonlinear σ-model or the 1d spin-chin of SYM composite operators. The physical quantities such as space-time correlation functions are derived from this S-matrix using various integrability techniques.

1. Asymptotic Bethe ansatz

We first consider composite operators in the $su(2)$ sector with only one type of impurities $X \equiv \Phi_{1\dot{1}}$, like $\mathrm{Tr}[Z\cdots ZX\cdots XZ\cdots X\cdots]$. Composite operators are mixed to each other under the dilatation. At one-loop order, the mixing matrix is given by

$$\Gamma = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left(1 - \mathcal{P}_{l,l+1}\right)\,, \qquad (22)$$

which is nothing but Heisenberg quantum spin-chain model [45] where $\mathcal{P}$ is the permutation operator on two spin states $|\Uparrow\rangle \equiv Z$ and $|\Downarrow\rangle \equiv X$. The "magnon" excitations X's on the spin chain are dual to string states called "giant magnon" in S^2 subspace of S^5 [27]. The conformal dimensions are given by eigenvalues of this "integrable" spin-chain Hamiltonian, which are solved by Bethe ansatz equations (BAEs) [46]. It is possible to continue this procedure to a few next orders but impossible to generalize to arbitrarily higher ones.

Another way to view this is to consider a state with N number of $\Phi_{1\dot{1}}$ with momenta $\{p_i\}$

$$\sum_{1\le x_1<\cdots<x_N\le L} e^{i(p_1x_1+\cdots+p_Nx_N)}\mathrm{Tr}[Z\cdots \overset{x_1}{\overset{\downarrow}{X}}\cdots \overset{x_2}{\overset{\downarrow}{X}}\,Z\cdots \overset{x_N}{\overset{\downarrow}{X}}\cdots] = A_{1\dot{1}}^{\dagger}(p_1)A_{1\dot{1}}^{\dagger}(p_2)\cdots A_{1\dot{1}}^{\dagger}(p_N)|0\rangle.$$

One can consider a virtual process in which X with p_k moves in the right direction in the spin chain. While it moves past Z freely, it will scatter with other X's. From Eq. (12), one can see that scattering of two X's are diagonal and satisfy

$$A^\dagger_{1\dot{1}}(p_k)A^\dagger_{1\dot{1}}(p_j) = A(p_k,p_j)^2 A^\dagger_{1\dot{1}}(p_j)A^\dagger_{1\dot{1}}(p_k). \quad (23)$$

After consecutive scatterings with other X's and returning back to orginal position x_k (utilizing the cyclic property of trace), a periodicity condition or the "Bethe-Yang equation" [24,47] is obtained

$$\begin{aligned} 1 &= e^{ip_kL}\prod_{j\neq k}^{N} A(p_k,p_j)^2 \\ &= e^{ip_k(L-N)}e^{iP}\prod_{j\neq k}^{N} S_0^2(p_k,p_j)\left(\frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\right)^2, \end{aligned} \quad (24)$$

where $P = \sum_j p_j$. (Cyclicity restricts $P = 0$.) The weak coupling limit $g \ll 1$ of this equation coincides with the BAE of the Heisenberg spin-chain model. It is also consistent with string results in the $g \gg 1$ limit [48] and matches with conjectured all-loop BAEs [49].

For general states including all types of SYM fields, one should consider

$$\begin{aligned} &A^\dagger_{i_1\dot{i}_1}(p_1)A^\dagger_{i_2\dot{i}_2}(p_2)\cdots A^\dagger_{i_N\dot{i}_N}(p_N)|0\rangle, \\ &i_1,\dot{i}_1,\cdots,i_N,\dot{i}_N = 1,2,3,4. \end{aligned} \quad (25)$$

During the scattering process with each other, the $su(2|2)$ indices change since the S-matrix is non-diagonal as can be seen in Eq. (12). The Bethe-Yang equations can be derived by diagonalizing very complicted "transfer" matrix which acts on states like (25) [50,51]. This has been done by *nested* algebraic Bethe ansatz [52], by nested coordinate Bethe ansatz [18, 53] or by analytic Bethe ansatz [32]. The resulting BAEs are

$$\begin{aligned} &e^{ip_k(-L+N-\frac{m_1^{(1)}}{2}-\frac{m_1^{(2)}}{2})} = e^{iP}\prod_{j\neq k}^{N} S_0^2(p_k,p_j)\left(\frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\right)^2 \prod_{\alpha=1}^{2}\prod_{j=1}^{m_1^{(\alpha)}}\left(\frac{x_k^- - y_j^{(\alpha)}}{x_k^+ - y_j^{(\alpha)}}\right), \\ &e^{i\frac{P}{2}}\prod_{k=1}^{N}\left(\frac{y_j^{(\alpha)} - x_k^-}{y_j^{(\alpha)} - x_k^+}\right) = \prod_{l=1}^{m_2^{(\alpha)}}\left(\frac{v_j^{(\alpha)} - w_l^{(\alpha)} + \frac{i}{2g}}{v_j^{(\alpha)} - w_l^{(\alpha)} - \frac{i}{2g}}\right), \\ &\prod_{j=1}^{m_1^{(\alpha)}}\left(\frac{w_l^{(\alpha)} - v_j^{(\alpha)} + \frac{i}{2g}}{w_l^{(\alpha)} - v_j^{(\alpha)} - \frac{i}{2g}}\right) = \prod_{k\neq l}^{m_2^{(\alpha)}}\left(\frac{w_l^{(\alpha)} - w_k^{(\alpha)} + \frac{i}{g}}{w_l^{(\alpha)} - w_k^{(\alpha)} - \frac{i}{g}}\right), \end{aligned} \quad (26)$$

where $v_j^{(\alpha)} \equiv y_j^{(\alpha)} + 1/y_j^{(\alpha)}$. The anomalous dimension of a state is given by

$$\Gamma = 2ig\sum_{k=1}^{N}\left(\frac{1}{x_k^+} - \frac{1}{x_k^-}\right). \quad (27)$$

The $su(2)$ BAE, Eq. (24), is given by restricting all $m_i^{(\alpha)} = 0$. These AdS_5/CFT_4 BAEs derived from the S-matrix are identical to those already conjectured in [17].

2. Finite-size effects

All-loop asymptotic BAEs are derived from the S-matrix which is defined for asymptotic particle states. The distances between the particles should be much larger than Compton wave-length, or the size $L \gg 1/m$. If not, the aymptotic BAEs are not valid. However, it is the finte L which is interesting since it is either the size of composite operators or angular momentum of the string theory.

The S-matrix still plays an important role also even in this finite-size computation. Generalized Lüscher corrections [54] have applied to both strong [55,56] and weak coupling limits [57] of AdS/CFT computations where 4-loop anomalous dimension of the $su(2)$ Konishi operator matches with perturbative computations [58,59].

Standard method for finite-size effects in integrability is thermodynamic Bethe ansatz (TBA) [60, 61] or Y-system [62]. 2d space-time is viewed as Euclidean rectangular (in fact, a torus with periodicity imposed on each parallel edges) with finite spatial size L and infinite temporal size. If the channel duality (space↔time) is imposed, the theory is now defined on infinite spatial size where the S-matrix is well-defined but with finite temperature $1/L$ due to finite temporal size. The finite-size effect (Casimir energy) is computed from the free energy of the scattering particles with temperature $1/L$. At thermodynamic equilibrium, the free energy should be minimized with constraints that the momenta of particles should satisfy the asymptotic BAEs.

There are three nontrivial steps to follow in applying TBA to AdS/CFT. In space↔time, the Euclidean 2-vector (p, iE) maps to (iE, p) which should be identified with "mirror" 2-vector $(\tilde{p}, i\tilde{E})$, namely, $\tilde{p} = iE,\ \tilde{E} = -ip$. For a relativisitc case with a dispersion relation $E^2 - p^2 = m^2$, its mirror one is still given by $\tilde{E}^2 - \tilde{p}^2 = m^2$. However, the mirror dispersion relation derived from Eq. (15) by this mirror transformation is [63]

$$\tilde{E} = 2\sinh^{-1}\left(\frac{\sqrt{\tilde{p}^2+1}}{4g}\right). \tag{28}$$

The S-matrix and asymptotic BAEs (26) are basically the same as before with proper analytic continuations such as $p_k \to \tilde{p}_k,\ L \to R = \infty$. Next step is the bound states which occur as poles of the S-matrix [44]. In the infinite R limit, the Bethe strings are known to be exact solutions of the BAEs which describe the bound states. "Diagonalized" S-matrices between the bound states can be constructed easily from the "string hypothesis" [64]. The most general Bethe-Yang equations in the mirror space can be written including these infinite number of bound states. The TBA equations can be derived by standard procedure as infinte number of coupled non-linear integral equations [65–69]. They can be transformed to "Y-systems" which are functional difference equations:

$$Y_{N,M}\left(u+\frac{i}{2g}\right)Y_{N,M}\left(u-\frac{i}{2g}\right) = \frac{[1+Y_{N,M+1}(u)][1+Y_{N,M-1}(u)]}{[1+Y^{-1}_{N-1,M}(u)][1+Y^{-1}_{N+1,M}(u)]}, \tag{29}$$

where the indices (N, M) cover 2D integral lattices

$$(N,M) = \begin{cases}(n,\nu) & \text{with} \quad \nu=+1,0,-1,\ n=2,\cdots,\infty \\ (1,\pm m) & \text{with} \quad m=2,\cdots,\infty \\ (1,1), & (1,-1),\ (2,2),\ (2,-2)\,.\end{cases} \tag{30}$$

The last step is to generalize the TBA of the vacuum state to those for excited states which include some excited particles in the physical space [70–72]. This can be done either by analytic continuations following [73] or by using the above Y-systems (29) with analytic properties associated with large L asymptotic conditions. The energy formula is given by

$$\mathcal{E}(L) = \sum_k E(p_k) - \sum_{Q=1}^{\infty}\int \frac{du}{2\pi}\partial_u \tilde{p}_Q \log(1+Y_{Q,0}) \tag{31}$$

where $\tilde{p}_Q$ is defined by (28) with 1 replaced by Q^2.

The analytic and numerical studies of the TBA equations [74–78] show good agreements with string theory results [79–83] and with 5-loop Lüscher corrections [84, 85]. To simplify TBA system which contain infinite number of coupled integral equations, finite number of non-linear integral equations (NLIE) [86, 87] and quantum spectral curve formalism have been proposed [88].

3. Structure constants

We have so far described how to determine exact conformal dimensions of the single-trace composite operators in the planar $\mathcal{N} = 4$ SYM theory. The conformal dimensions fix the two-point correlation functions with proper normalizations. Another important quantity is three-point correlation functions

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{C_{ijk}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}}\,, \tag{32}$$

where the coefficients C_{ijk} are the structure constants. These constants define OPE expansions of two local operators and general multi-point functions in a conformal field theory can be constructed using conformal bootstraps, at least in principle. There have been several interesting approaches to apply the integrability found in the spectral problem to the structure constants in both gauge [89] and string theory sides [90–92] of AdS/CFT. One of recent efforts is to compute "form factors", which are matrix elements of local operators between two asymptotic states, $C_{ijk} = {}_{\text{out}}\langle \mathcal{O}_i | V_{\mathcal{O}_k} | \mathcal{O}_j\rangle_{\text{in}}$ [93,94]. However, it is not clear how to find the vertex operator $V_{\mathcal{O}_k}$ and if its exact form factors can be completely determined by the S-matrix. Another proposal is to cut the string three-point vertex into two hexagon amplitudes and to sum over all possible internal states locating on three edges which have been cut and sum over all distributions of the asymptotic particles in the three operators to the two hexagons [95]. While this proposal is promising in that the hexagon amplitudes for a given particle configuration is completely determined by the $su(2|2)$ S-matrix (12), main technical difficulties arise in the sums. This proposal has been checked in various simple cases.

IV. OTHER INTEGRABLE S-MATRICES

Integrable deformations and 3d AdS/CFT are briefly reviewed.

1. S-matrices of Deformed AdS/CFTs

A. β-deformation

This is one-parameter marginal deformation, obtained by replacing the superpotential for the chiral superfields of the original $\mathcal{N}=4$ SYM by

$$W = ih\ \mathrm{tr}(e^{i\pi\beta}\phi\psi Z - e^{-i\pi\beta}\phi Z\psi). \tag{33}$$

The deformation breaks the supersymmetry down to $\mathcal{N}=1$ but still maintains the conformal invariance in the planar limit to all perturbative orders [96,97], since the deformation becomes exactly marginal with $h\bar{h} = g_{\mathrm{YM}}^2$. When β is real, this deformed SYM theory is dual to a type-IIB string theory on the Lunin-Maldacena background [98], which is obtained by a so-called TsT transformation.

In the weak coupling limit, various perturbative analysis of the deformed SYM has been studied [99] and, in particular, anomalous dimensions for the one and two magnon states in the $su(2)$ sector have been computed up to four loops [100]. There have been several indications that the anomalous dimensions of the β-deformed SYM are exactly solvable. Perturbative dilatation operators are mapped to some integrable spin chains [101]. Exact S-matrix has been proposed as a transform of the original $su(2|2)$-invariant S-matrix, [102]

$$\tilde{S}(p_1, p_2) = F \cdot S(p_1, p_2) \cdot F, \tag{34}$$

where F is a Drinfeld-Reshetikhin twist matrix given by

$$F = e^{2\pi i\beta(h\otimes\mathbb{I}\otimes\mathbb{I}\otimes h - \mathbb{I}\otimes h\otimes h\otimes\mathbb{I})},$$
$$h = \mathrm{diag}\left(\frac{1}{2}, -\frac{1}{2}, 0, 0\right). \tag{35}$$

By construction, the Yang-Baxter equation is satisfied. With this S-matrix along with some twisted boundary conditions, conjectured all-loop Bethe ansatz equations have been confirmed [103]. The perturbative four-loop anomalous dimension of the Konishi operator in the deformed gauge theory is reproduced from the Lüscher formula [104]. (Similar results can be obtained by introducing "operatorial" twisted boundary conditions [105] or Y-system with twisted asymptotic conditions [106].) Also in the strong coupling limit, consistency of the twisted S-matrix has been shown in the finite-size effects of string configurations and world-sheet scattering amplitudes.

The proposed S-matrix is also valid for $\mathcal{N}=4$ SYM deformed by three parameters with proper boundary twists [102]. While next-to-leading order wrapping corrections of the vacuum energy has been computed [107], this deformed theory may not be exactly conformal as claimed recently [108].

B. η-deformation

It is possible to construct a S-matrix which is symmetric under centrally extended $su(2|2)_q$, quantum deformation of universal enveloping algebra, simply quantum group. As before, the symmetry algebra determines the S-matrix elements. The matrix structure is similar as Eq. (12), but individual amplitudes become more complicated and not presented here [109]. The energy-momentum dispersion relation is also deformed to

$$\left(1 - g^2(q-q^{-1})^2\right)\left[\frac{E}{2}\right]_q^2 - 4g^2\sin^2\frac{p}{2} = \left[\frac{1}{2}\right]_q^2, \tag{36}$$

with 'q-number' defined by $[n]_q \equiv (q^n - q^{-n})/(q-q^{-1})$. The limit $q\to 1$ reproduces the original $su(2|2)$ S-matrix and dispersion relation (15).

There are several evidences that this S-matrix describes scatterings on the world-sheet which string action is a nonlinear sigma-model with integrable η-deformations related to solutions of classical Yang-Baxter equation [110, 111]. The scattering amplitudes based on this string action in the $g \gg 1$ limit are shown to be consistent with the $su(2|2)_q$ S-matrix for real $q = e^{-\nu/g}$, $\nu = 2\eta/(1+\eta^2)$ [112]. Its dual gauge theory is not much understood while it is believed that the deformation q should be related to 4d space-time non-commutativity.

2. S-matrices for $\mathrm{AdS}_4/\mathrm{CFT}_3$

Aharony-Bergman-Jafferis-Maldacena (ABJM) model is 3d $\mathcal{N}=6$ superconformal Chern-Simons guage theory with matter fields [113]. It is a quiver gauge theory with an action

$$S = \frac{k}{4\pi}\int d^3x \quad \mathrm{Tr}\Big[\epsilon^{\mu\nu\lambda}\left(A_\mu\partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda - \hat{A}_\mu\partial_\nu\hat{A}_\lambda - \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\right) \\ -(\mathcal{D}_\mu Y)^\dagger\mathcal{D}^\mu Y - i\Psi^\dagger \not{D}\Psi - V_{\mathrm{ferm}} - V_{\mathrm{bos}}\Big]\,, \tag{37}$$

where V_{bos} and V_{ferm} are the sextic bosonic and quartic mixed potentials. The gauge fields A_μ ($\hat{A}_\mu$) are adjoints of $U(N)$ ($\hat{U}(N)$) gauge groups, and non-dynamical. Matter hypermultiplets Y^A, Ψ_A ($A = 1, 2, 3, 4$) are bi-fundamentals $(N, \bar{N})$ representations and $Y_A^\dagger$, $\Psi^{\dagger A}$ are anti-bifundamentals in $(\bar{N}, N)$. This ABJM theory has a full symmetry group OSp(6|4).

Single-trace composite operators are alternating spin-chains of bifundamentals and anti-bifundamentals. The BPS vacuum operator is [114,115]

$$\mathrm{Tr}\left(Y^1 Y_4^\dagger Y^1 Y_4^\dagger \cdots Y^1 Y_4^\dagger\right) \equiv |0\rangle. \tag{38}$$

General composite operators contain two types of excitations, "A-particles" $(Y^2, Y^3|\Psi_{4+}, \Psi_{4-})$ replacing Y^1 on the odd sites and "B-particles" $(Y_3^\dagger, Y_2^\dagger|\Psi_+^{\dagger 1}, \Psi_-^{\dagger 1})$ replacing $Y_4^\dagger$ on the even sites. Both belong to fundamental representations of $su(2|2)$. In the planar limit of $N,\ k \to \infty$ with $N/k \equiv \lambda$ fixed, leading two-loop dilatation operator of $su(4)$ sector of scalar fields is identified with integrable spin-chain Hamiltonian [116, 117]. The planar ABJM model is also dual to type IIA string theory moving on $\mathrm{AdS}_4 \times \mathrm{CP}^3$ backgound, which shows also classical integrability [118,119].

Motivated by these developments, non-perturbative integrability has been conjectured. If $A_i^\dagger$ and $B_i^\dagger$ denote A- and B-particle creation operators, respectively, multi-particle scattering states can be written as

$$| \overset{1}{\downarrow}{Y^1}\, Y_4^\dagger \cdots \overset{x_1}{\downarrow}{\chi_{A_1}}\, Y_4^\dagger \cdots Y^1\, \overset{x_M}{\downarrow}{\chi_{B_M}} \cdots Y^1\, \overset{2L}{\downarrow}{Y_4^\dagger}\rangle \\ = A_{\chi_1}^\dagger(x_1)\cdots B_{\chi_M}^\dagger(x_M)|0\rangle. \tag{39}$$

The integrability is realized by factorized S-matrices in the momentum space. Since there are two sets of $su(2|2)$ excitations, four combinations of scatterings are defined by

$$\mathbb{X}_i^\dagger(p_1)\,\mathbb{Y}_j^\dagger(p_2)|0\rangle_{\mathrm{in}} \\ = \sum_{i',j'=1}^{4} S^{(\mathbb{X}\mathbb{Y})}{}_{i\,j}^{i'j'}(p_1,p_2)\,\mathbb{Y}_{j'}^\dagger(p_2)\,\mathbb{X}_{i'}^\dagger(p_1)|0\rangle_{\mathrm{out}}\,, \\ \mathbb{X},\mathbb{Y} = A, B. \tag{40}$$

Notice that there is no reflected "out" states $\mathbb{X}_{j'}^\dagger(p_2)\,\mathbb{Y}_{i'}^\dagger(p_1)|0\rangle_{\mathrm{out}}$ in the RHS of (40). This reflectionlessness is a basic assumption which can be justified by other analysis [120]. The charge conjugation and $su(2|2)$ symmetries determine the S-matrices [121]

$$S^{(AA)}(p_1,p_2) = S^{(BB)}(p_1,p_2) = S_0(p_1,p_2)\widehat{S}(p_1,p_2)\,, \\ S^{(AB)}(p_1,p_2) = S^{(BA)}(p_1,p_2) = \tilde{S}_0(p_1,p_2)\widehat{S}(p_1,p_2)\,, \tag{41}$$

where $\widehat{S}$ is given by omitting S_0 from Eqs.(12) and (13). The scalar factors can be determined by unitarity and crossing [121]

$$S_0(p_1,p_2) = \frac{1-\frac{1}{x_1^+x_2^-}}{1-\frac{1}{x_1^-x_2^+}}\sigma(p_1,p_2), \\ \tilde{S}_0(p_1,p_2) = \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\sigma(p_1,p_2), \tag{42}$$

with the BES phase factor $\sigma(p_1, p_2)$ in Eq.(20). The dispersion relation of these particles is given by

$$E = \sqrt{\frac{1}{4} + 4h(\lambda)^2\sin^2\frac{p}{2}}\,. \tag{43}$$

The interpolating function $h(\lambda)$ is not completely fixed by the symmetry.

This S-matrix reproduces the all-loop BAEs conjectured earlier [122]. It has been also confirmed by spin-chain BAE computations in the weak coupling [123] and by scattering amplitudes on the world-sheet in the strong coupling [120]. TBA, Y-systems, and quantum spectral curve for finite-size effects have been also constructed [124–126]. The analysis based on these equations lead to a conjecture on $h(\lambda)$ by comparing the localization computation [127]:

$$\lambda = \frac{\sinh(2\pi h)}{2\pi}\ {}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,\frac{3}{2};-\sinh^2(2\pi h)\right). \tag{44}$$

This conjecture matches with all existing perturbative results in both weak and strong coupling limits.

V. CONCLUDING REMARKS

Integrability in AdS/CFT is a rapidly developing subject which covers wide areas of gauge and string theories in truly non-perturbative and quantitative way. In this short review, we have focused on the world-sheet S-matrices and briefly explained the applications. We recommend to read original references for details.

Several important issues related to S-matrices could not be covered. Integrability and S-matrix for the string dynamics in the strong coupling have been left out. Another topic is boundary integrability. Open strings attached on giant gravitons, which are D3-branes wrapping 3-cycles in the S^5, are dual to composite operators which are baryon-like determinant operators in $\mathcal{N} = 4$ SYM theory:

$$\epsilon_{i_1\cdots i_N}\epsilon^{j_1\cdots j_N}\Phi^{i_1}_{j_1}\cdots\Phi^{i_{N-1}}_{j_{N-1}}(Z\cdots Z\chi Z\cdots\chi' Z\cdots Z)^{i_N}_{j_N},\quad \Phi = Z,\ Y\,. \tag{45}$$

The integrable boundary introduces boundary scattering matrices, which can be completely determined by the symmetry [128]. These boundary S-matrices satisfy the boundary Yang-Baxter equations and are used to construct boundary asymptotic BAEs. Another development related to the boundary problem is to compute expectation values of cusped Wilson loop operators. By inserting scattering states at the location of the cusp, the each half-line of the non-BPS Wilson loop operator becomes both boundaries. The symmetry of the excitations is reduced from $su(2|2)\otimes su(2|2)$ to $su(2|2)$ which decides the boundary S-matrix to be the same as the single factor of the bulk S-matrix (12) [129,130]. TBA derived from these S-matrices has determined the expectation values exactly. We have also omitted recent developments of exact S-matrices of AdS_3/CFT_2, which are $su(1|1)$ symmetry. We refer to a recent review [131] for this subject.

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