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RG flows of non-unitary minimal CFTs

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In this paper we study the renormalization group flow of the (p, q) minimal (non-unitary) CFT perturbed by the $\Phi_{1,3}$ operator with a positive coupling. In the perturbative region $q \gg (q-p)$, we find a new IR fixed point which corresponds to the $(2p-q, p)$ minimal CFT. The perturbing field near the new IR fixed point is identified with the irrelevant $\Phi_{3,1}$ operator. We extend this result to show that the non-diagonal ((A, D)-type) modular invariant partition function of the (p, q) minimal CFT flows into the (A, D)-type partition function of the $(2p-q, p)$ minimal CFT and the diagonal partition function into the diagonal.

1. Conformal field theories (CFTs) [1] perturbed by relevant operators have provided a theoretical framework for constructing integrable scale non-invariant 2D quantum field theories (QFTs). There are two interesting classes: One is a class of massive integrable QFTs whose scattering matrices are exactly solvable due to an infinite number of conserved currents [2]. The other is a class of scale non-invariant QFTs with no massive particles which have new RG fixed points at which the scale invariance and conformal symmetries are restored [3,4].

The minimal CFTs [1] are characterized by two coprime integers p, q $(q>p)$. We will denote the (p, q) minimal CFT by $\mathcal{M}_{(p,q)}$. The principal series of $(p, p+1)$ correspond to the unitary CFTs in the sense that the states created by the Virasoro generators have positive definite norm [5]. Except for the string theory, the unitarity of these Virasoro modules seems not the first principle to be satisfied. Interesting applications of the non-unitary CFTs have been made in the integrable lattice models, matrix models and others.

We start with the following perturbed CFT:

$$\mathcal{S}_{(p,q)}(g)=\mathcal{M}_{(p,q)}+g\int \mathrm{d}^2z\, \Phi_{1,3}(z,\bar{z}) \,, \tag{1}$$

where the dimension of the least relevant operator $\Phi_{1,3}$ of $\mathcal{M}_{(p,q)}$ is

$$\Delta(\Phi_{1,3})=1-\frac{2(q-p)}{q} \,. \tag{2}$$

The theories can be represented in one-dimensional parameter space spanned by g because $\Phi_{1,3}$ satisfies a closed operator product expansion,

$$[\Phi_{1,3}]\times[\Phi_{1,3}]=[\Phi_{1,3}] \,. \tag{3}$$

If $g<0$, the theory is a massive integrable QFT, the "restricted sine-Gordon theory" both for the unitary CFTs [6] and the non-unitary CFTs [7]. If $g>0$, the perturbed theory remains as a massless theory while the scale invariance is broken. For the unitary CFT $\mathcal{M}_{(p,p+1)}$, Zamolodchikov [3] and Ludwig and Cardy [4] found a new IR fixed point corresponding to another unitary theory $\mathcal{M}_{(p-1,p)}$ for $\Delta(\Phi_{1,3})=1-\epsilon$ with $\epsilon=2/(p+1)\ll 1$. The perturbing field $\Phi_{1,3}$ becomes irrelevant field $\Phi_{3,1}$ near the new fixed point.

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In this paper, we study the RG flows of the non-unitary CFTs $\mathcal{M}_{(p,q)}$ to show that $\mathcal{M}_{(p,q)}$ perturbed by $\Phi_{1,3}$ flows into $\mathcal{M}_{(2p-q,p)}$ perturbed by $\Phi_{3,1}$ for $q \gg (q-p)$. This is an extension of the unitary CFTs [3,4,9] and has been noticed in a different context in ref. [10]. We further analyze the RG flows of the non-diagonal modular invariant partition functions (MIPFs) of $\mathcal{M}_{(p,q)}$ [8] to find that the (A, A) and (A, D) MIPFs flow into the (A, A) and (A, D) MIPFs, respectively. This result is consistent with some early results on the RG flows of the unitary CFTs in refs. [11,12].

2. Under the scale transformation $x_\mu \to (1+1/2\mathrm{d}t)x_\mu$, the parameter g changes according to the equation [3]

$$\beta(g)=\frac{\mathrm{d}g}{\mathrm{d}t}=\epsilon g-\tfrac{1}{2}Cg^2+\mathrm{O}(g^3)\,,\quad \epsilon=\frac{2(q-p)}{q}\,, \tag{4}$$

where the coefficient of a (diagonal) three-point function $C=C_{(1,3)(1,3)(1,3)}$ is [13]

$$C_{(1,3)(r,s)(r,s)}=-\frac{\Gamma(\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{\gamma^3(1-\epsilon/2)}{\gamma(2-3\epsilon/2)}\right)^{1/2}\frac{\gamma(r+1-s-(1-s)\epsilon)}{\gamma(r-s+(1+s)\epsilon)}\,, \tag{5}$$

with $\gamma(x)=\Gamma(x)/\Gamma(1-x)$. For $\epsilon \ll 1$, $C\simeq 4/\sqrt{3}+\mathrm{O}(\epsilon)$.

From eq. (4), one can find a new fixed point at $g=g_*$,

$$\beta(g_*)=0\to g_*\simeq\frac{\sqrt{3}}{2}\frac{2(q-p)}{q}\,. \tag{6}$$

One can identify this as the IR fixed point from the fact that $g_*=g(t_*)$ with $t_*=\infty$ from eq. (4). Following ref. [3], we introduce the 'c-function' which gives the central charges at the fixed points and satisfies the following RG equation:

$$\frac{\mathrm{d}c}{\mathrm{d}t}=\beta(g)\frac{\partial}{\partial g}c(g)=-12\beta^2(g)\,. \tag{7}$$

Eq. (7) shows that the c-function is monotonically decreasing as t increases in the $\Phi_{1,3}$ direction such that the inequality $c_{\mathrm{UV}}>c_{\mathrm{IR}}$ holds [#1]. From eqs. (4) and (7), the central charge of the IR CFT is given by

$$c_{\mathrm{IR}}=c(g_*)=c_{\mathrm{UV}}-6\epsilon g_*^2+2Cg_*^3+\mathrm{O}(g_*^4)\,,\quad c_{\mathrm{UV}}=1-\frac{6(q-p)^2}{pq}=1-\frac{6(q-p)^2}{p(2p-q)}=c[\mathcal{M}_{(2p-q,p)}]\,. \tag{8}$$

The conformal dimension of the perturbing field near the new fixed point is determined by

$$\Delta(\Phi_{1,3})=1-\left.\frac{\partial\beta}{\partial g}\right|_{g^*}=1+\epsilon+\epsilon^2+\mathrm{O}(\epsilon^3)\simeq\frac{q}{2p-q}=\Delta[\Phi_{3,1}]\quad\text{for } \mathcal{M}_{(2p-q,p)}\,. \tag{9}$$

This completes the RG flow of $\mathcal{M}_{(p,q)}$ perturbed by $\Phi_{1,3}$ into $\mathcal{M}_{(2p-q,p)}$ perturbed by $\Phi_{3,1}$ for $q\gg(q-p)$. This means that the (p,q) minimal CFTs can be grouped into an infinite number of series for each value of $q-p$ such that the CFTs in each series are connected by the RG flows under the $\Phi_{1,3}$ perturbations:

$$\ldots\to(p+n,p+2n)\to(p,p+n)\to(p-n,p)\to(p-2n,p-n)\to\ldots$$

for $p\gg n$ and $n=1, 2, \ldots$. The $n=1$ case gives the RG flows of the principal series (the unitary CFTs). No unitary CFTs flow into the non-unitary ones and vice versa.

3. We now consider how each MIPF of the minimal CFTs will flow under the perturbation. There are three

[#1] This does not contradict to the fact that "c-theorem" [3] is not valid for the non-unitary CFTs. Although the c-function is not monotonically decreasing for *any* relevant perturbation, there can still exist certain perturbations which makes c decrease.

types of the MIPFs for the (p, q) minimal CFT, denoted by $Z_{(p,q)}$ (A, A), $Z_{(p,q)}$ (A, D), and $Z_{(p,q)}$ (A, E) [8]. The diagonal MIPFs $Z_{(p,q)}$ (A, A) are possible for all (p, q) and contain only spinless fields. The non-diagonal MIPFs $Z_{(p,q)}$ (A, D) are possible for p or q even and included some primary fields with spins. The exceptional MIPFs $Z_{(p,q)}$ (A, E) are allowed only for a few discrete values of p and q. We will consider the RG flows of the $Z_{(p,q)}$ (A, A) and $Z_{(p,q)}$ (A, D) MIPFs here.

The partition functions of the perturbed (p, q) minimal CFTs on the RG trajectory can be written in the following schematic form:

$$Z(g)=\int [\mathscr{D}\varphi]\,\exp\{-\mathscr{S}_{(p,q)}(g)[\varphi]\}=\int [\mathscr{D}\varphi]\,\exp\left[-\left(\mathscr{M}_{p/q}+g\int \mathrm{d}^2z\,\Phi_{1,3}(z,\bar{z})\right)\right], \tag{10}$$

where φ denotes the field degree of freedom which has no direct consequence in the following context. For a small $g_* \ll 1$, eq. (10) gives with $Z(0)=Z_{(p,q)}$ and $Z(g_*)=Z_{(2p-q,p)}$,

$$\delta Z=Z_{(2p-q,p)}(\mathrm{A},Y)-Z_{(p,q)}(\mathrm{A},X)\simeq -g_*\frac{\tau_2}{\pi}\langle \Phi_{1,3}\rangle_{\mathrm{torus}}, \tag{11}$$

where τ_2 is the imaginary part of the modular parameter τ. For a given MIPF of $\mathscr{M}_{(p,q)}$ (X=A or D), we need to find the corresponding Y for the MIPFs of $\mathscr{M}_{(2p-q,p)}$. We will show that Y=A if X=A and Y=D if X=D using eq. (11).

For the purpose, it is convenient to express the MIPFs of the minimal CFTs in terms of the Coulomb gas form [14]:

$$Z_{(p,q)}(\mathrm{A},\mathrm{A})=\frac{1}{2}\left[Z_{\mathrm{c}}(pq)-Z_{\mathrm{c}}\left(\frac{p}{q}\right)\right],$$

$$Z_{(p,q)}(\mathrm{A},\mathrm{D})=\frac{1}{2}\left[Z_{\mathrm{c}}\left(\frac{4p}{q}\right)-Z_{\mathrm{c}}\left(\frac{p}{q}\right)-Z_{\mathrm{c}}\left(\frac{4}{pq}\right)+Z_{\mathrm{c}}\left(\frac{1}{pq}\right)\right],$$

$$Z_{\mathrm{c}}(N)=\frac{1}{|\eta(\tau)|^2}\sum_{e,m\in \mathbf{Z}}\exp[\tfrac{1}{2}\pi i\tau(e/\sqrt{N}+m\sqrt{N})^2]\,\exp[-\tfrac{1}{2}\pi i\bar{\tau}(e/\sqrt{N}-m\sqrt{N})^2]\,. \tag{12}$$

We first compute δZ for the diagonal and non-diagonal MIPFs separately assuming that (A, A) flows into (A, A) and (A, D) to (A, D):

$$\delta_{\mathrm{A}}Z=Z_{(2p-q,p)}(\mathrm{A},\mathrm{A})-Z_{(p,q)}(\mathrm{A},\mathrm{A})\simeq 2\pi\tau_2\frac{(q-p)}{q}\frac{1}{|\eta(\tau)|^2}F(pq)+\mathrm{O}(\epsilon^2),$$

$$\delta_{\mathrm{D}}Z=Z_{(2p-q,p)}(\mathrm{A},\mathrm{D})-Z_{(p,q)}(\mathrm{A},\mathrm{D})\simeq 2\pi\tau_2\frac{(q-p)}{q}\frac{1}{|\eta(\tau)|^2}[F(pq)-F(pq/4)]+\mathrm{O}(\epsilon^2), \tag{13}$$

using $Z_{\mathrm{C}}(N)=Z_{\mathrm{C}}(1/N)$ and

$$\delta Z_{\mathrm{C}}(N)=Z_{\mathrm{C}}(N+\delta N)-Z_{\mathrm{C}}(N)\simeq -2\pi\tau_2\frac{\delta N}{N}\frac{1}{|\eta(\tau)|^2}F(N),$$

$$F(N)=\sum_{e,m\in\mathbf{Z}}\left(\frac{m^2N}{2}-\frac{e^2}{2N}\right)\exp[\tfrac{1}{2}\pi i\tau(e/\sqrt{N}+m\sqrt{N})^2]\,\exp[-\tfrac{1}{2}\pi i\bar{\tau}(e/\sqrt{N}-m\sqrt{N})^2]\,. \tag{14}$$

From the asymptotic expression of $F(N)$

$$F(N)\sim -\frac{1}{2\pi\tau_2^{2/3}}\frac{\sqrt{N}}{2}\quad \text{for } N\gg 1, \tag{15}$$

one can find $\delta_A Z$ and $\delta_D Z$ to be

$$\delta_A Z \simeq -\frac{(q-p)}{2\sqrt{\tau_2}}\frac{1}{|\eta(\tau)|^2}, \quad \delta_D Z \simeq -\frac{(q-p)}{4\sqrt{\tau_2}}\frac{1}{|\eta(\tau)|^2}. \tag{16}$$

Next we compute the one-point function on the torus using the method used in refs. [9,10]. Since $\Delta[\Phi_{1,3}] \simeq 1$, one can express the one-point function with the characters

$$\langle \Phi_{1,3} \rangle_{\text{torus}}(\tau, \bar{\tau}) = 4\pi |z|^2 \sum_{F\in\mathcal{A}} \langle \Phi_F | \Phi_{1,3}(z, \bar{z}) | \Phi_F \rangle \chi_h(\tau) \bar{\chi}_{\bar{h}}(\bar{\tau}) = 4\pi |z|^2 \sum_{F\in\mathcal{A}} C_{(1,3),F,F} \chi_h(\tau) \bar{\chi}_{\bar{h}}(\bar{\tau}), \tag{17}$$

where $\mathcal{A}$ denotes a complete set of primary fields F with conformal dimension $(h, \bar{h})$. The characters are expressed as a sum of infinite terms as follows:

$$\chi_{r,s}(\tau) = \frac{K_{r,s}(\tau) - K_{r,-s}(\tau)}{\eta(\tau)}, \quad K_{r,s}(\tau) = \sum_{n=-\infty}^{\infty} \exp\left(2\pi i \tau \frac{(2pqn + qr - ps)^2}{4pq}\right). \tag{18}$$

Since p, $q \gg 1$ and $\tau_2 > 0$, most terms in $K_{r,s}$ vanish except those with $n=0$ and $qr - ps \sim p$, q. By the same reason, $K_{r,-s}$ becomes negligible. Therefore, the character can be approximated as

$$\chi_{r,s}(\tau) \simeq \frac{1}{\eta(\tau)} \exp\left(\pi i \tau \frac{(qr-ps)^2}{2pq}\right). \tag{19}$$

The integer $\lambda = qr - ps$ covers all integers between $-pq$ and pq only once for $1 \leqslant r \leqslant p$ and $1 \leqslant s \leqslant q$ for (A, A) and all the odd integers for (A, D). Furthermore, as pointed out in ref. [10], the non-diagonal combinations like $\chi_{r,s}\bar{\chi}_{r,q-s}$ are negligible because $|qr - p(q-s)| \sim pq$ if $|qr - ps| \sim O(1)$.

Using these observations and the coefficients of the diagonal three-point functions in eq. (5), one can obtain the one-point function on the torus to be

$$\begin{aligned}\langle \Phi_{1,3} \rangle_{\text{torus}} &\simeq -\alpha \frac{\pi^2}{\sqrt{3}|\eta|^2} \sum_{\lambda=-q}^{q} \frac{\Gamma((p+\lambda)/q)\Gamma((p-\lambda)/q)}{\Gamma((q-p+\lambda)/q)\Gamma((q-p-\lambda)/q)} \exp\left(-\pi\tau_2 \frac{\lambda^2}{pq}\right) + O(\epsilon) \\ &\simeq \alpha \frac{q\pi^2}{\sqrt{3}|\eta|^2} \int_{-\infty}^{\infty} dx\, x^2 \exp(-\pi\tau_2 x^2) = \frac{\alpha}{|\eta|^2} \frac{q\pi}{2\sqrt{3}\tau_2^{3/2}},\end{aligned} \tag{20}$$

where $\alpha = 1$ or $\frac{1}{2}$ for the (A, A) and (A, D) MIPFs, respectively. Comparing this with eq. (6), one can confirm eq. (16) with $X = Y = \text{A}$ and $X = Y = \text{D}$:

$$\mathcal{M}_{(p,q)}(\text{A, A}) \to \mathcal{M}_{(2p-q,p)}(\text{A, A}), \quad \mathcal{M}_{(p,q)}(\text{A, D}) \to \mathcal{M}_{(2p-q,p)}(\text{A, D}). \tag{21}$$

4. We showed so far that there can exist a RG flow from the (p,q) minimal CFT to the $(2p-q, p)$ minimal CFT due to the least relevant operator $\Phi_{1,3}$. Although our argument is rigorous only for $q \gg (q-p)$, it may be possible to extend our results to all possible pairs of (p,q). For the unitary CFTs, in particular, Zamolodchikov showed the RG flows for any p using the conjectured thermodynamic Bethe Ansatz (TBA) equations [15]. For example, the RG flow connects the tricritical Ising model with the Ising model. Although the direct derivation of the TBA equations is still missing, the TBA analysis can support the conjecture that the RG flow exists for all $(p, p+1)$ unitary CFTs. The interesting point is that the conjectured TBA equations for the massless $g > 0$ field theories are given by those for the massive $g < 0$ theories, i.e. the restricted sine-Gordon theories with the massless left- and right-moving particles instead of the massive particles. It would be interesting to study the RG flows of the (p, q) minimal CFTs in the same way as the unitary cases. The conjectured TBA equations will be provided by those for the massive theories [16,17]. This TBA analysis of the RG flow of the minimal CFTs based on the TBA equations will be reported elsewhere [18].

One interesting application of the RG flow is to identify a new integrable model which flows into the "Yang–Lee edge singularity model", identified with $\mathcal{M}_{(2,5)}$ [19]. This new model which is $\mathcal{M}_{(5,8)}$ can be realized as an UV (short distance) limit of the Yang–Lee model perturbed by the dimension-four operator $T_4(z)$ which is a decendent field of the vacuum at level 4:

$$S_{(5,8)} = S_{\mathrm{YL}} + g \int \mathrm{d}^2 z\, T_4(z) \bar{T}_4(\bar{z}) . \tag{22}$$

Finally, one can think of the RG flows of the SU(2) coset CFTs which have extended symmetries like the superconformal invariance. For the unitary CFTs, it has been claimed that there exist the RG flows into new IR fixed points by the perturbation of the least relevant operator [20]. Based on this observation, we can conjecture the following RG flows of the SU(2) coset non-unitary CFTs due to the least relevant operator:

$$\frac{\mathrm{SU}(2)_K \times \mathrm{SU}(2)_L}{\mathrm{SU}(2)_{K+L}} + g\Phi_{\mathrm{pert}} \to \frac{\mathrm{SU}(2)_K \times \mathrm{SU}(2)_{L-K}}{\mathrm{SU}(2)_L} + g' \Phi'_{\mathrm{pert}} , \tag{23}$$

where $L+2 = p/(q-p)$ and $\Delta(\Phi_{\mathrm{pert}}) = (K+L)/(K+L+2)$ and $\Delta(\Phi'_{\mathrm{pert}}) = (K+L+2)/(K+L)$. Also, interesting is to compare these massless field theories ($g>0$) with the massive theories ($g<0$) considered in ref. [21] where exact S-matrices and particle spectrum are proposed.

Note added. After finishing this paper, we received a preprint [22] which studies the RG flows of the diagonal non-unitary CFTs.

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