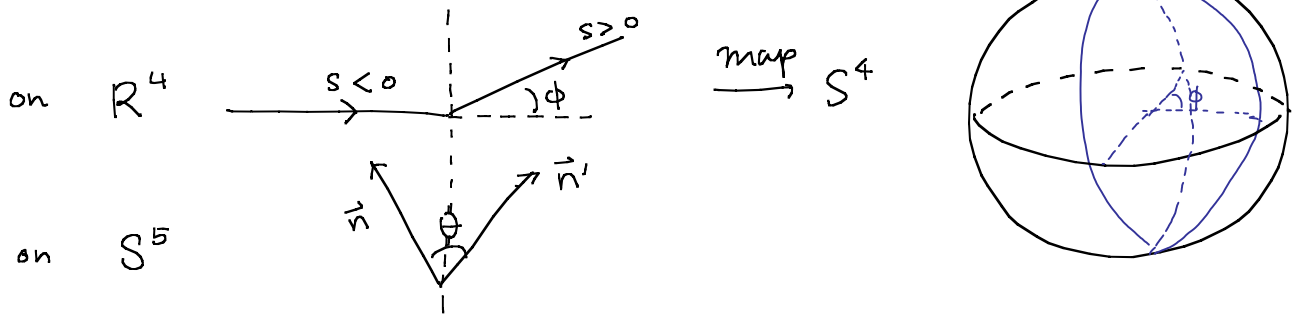


Exact results on WLs in N=4 SYM & ABJM

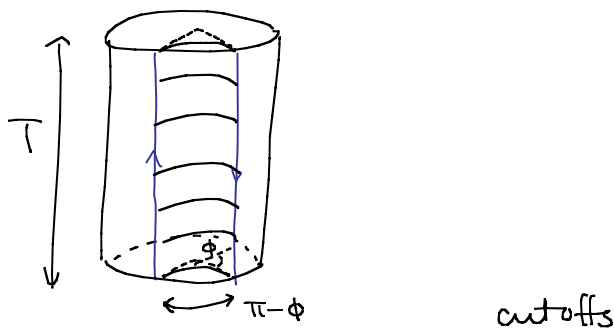
- Motivation: Nonperturbative is beyond imagination and can be understood only by results valid for any coupling in exact, quantitative way. ~~Pert~~, ~~semiclass.~~ ~~qualitative~~, ~~BS~~, ...
- AdS/CFT provides a promise for exact nonpert. results
- Integrability has been discovered for AdS_{d+1}/CFT_d d=3,4 and "spectrum problem" have been exactly solved.
- More exact solutions?
 - higher-point correlation functions } some progress but
 - space-time scattering amplitudes } very difficult

* Exact (any λ) results on \langle Wilson-Maldacena loops \rangle

Cusped WL:
$$W(\theta, \phi) = e^{\int ds (i A_\mu \cdot \dot{x}^\mu + \vec{n} \cdot \vec{\Phi} |\dot{x}|)}$$



Map: $\mathbb{R}^4 \rightarrow \mathbb{R} \times S^3$



$$\langle W_{\text{cusp}} \rangle = e^{-\Gamma_{\text{cusp}}(\lambda, \phi, \theta) \log \frac{L}{\epsilon}}$$

$$\langle W_{\text{Lines}} \rangle = e^{-T V_{\text{qf}}(\lambda, \theta, \phi)}$$

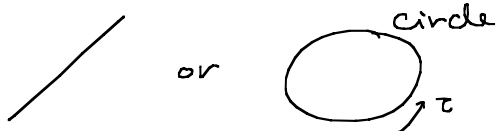
Physical observables

- ① $\phi \rightarrow i\varphi$: IR limit scatt. of mass particles, φ : boost
- ② Regge limit of 4-particle scatt. amplitude of $N=4$ SYM
($t \gg s$) $\log A \sim \log t \cdot \Gamma_{\text{cusp}}(\varphi)$
- ③ $\varphi \rightarrow \infty$, $\Gamma_{\text{cusp}} \sim \varphi \Gamma_{\text{cusp}}^\infty$: IR div of massless particles
- ④ quark- \bar{q} potential on S^3
- ⑤ $\phi \ll 1$, radiation from quark

Non perturbative (any λ) results on WL

① $\theta = \phi = 0$ [Circular WL] $\Gamma_{\text{cusp}} = 0$ $\frac{1}{2}$ BPS, $\langle W_0 \rangle(\lambda, N)$

① $\phi = 0$ (No cusp), $\theta \neq 0$



with $n_1 + i n_2 = \sin \theta e^{i\tau}$
 $n_3 = \cos \theta, n_4 = 0$

$\theta = \frac{\pi}{2}, \frac{1}{4}$ - BPS.

$\langle W_\theta \rangle(\lambda) = \langle W_0 \rangle(\lambda \cos^2 \theta)$

if $\theta \ll 1$: $\lambda' \cong \lambda(1 - \theta^2)$
 $\frac{\langle W_\theta \rangle - \langle W_{\theta=0} \rangle}{\langle W_{\theta=0} \rangle} \cong -\theta^2 \lambda \partial_\lambda \log \langle W_0 \rangle$

$$\theta^2 \frac{1}{2} \int_0^{2\pi} d\tau \int_0^{2\pi} dz' \underbrace{\hat{n}^i(z) \hat{n}^j(z')}_{\text{from } \hat{n}^1 \& \hat{n}^2} \langle \langle \Phi^i(z) \Phi^j(z') \rangle \rangle$$

$$\langle \langle \Phi(t) \Phi(\omega) \rangle \rangle = \frac{\gamma}{t^2}, \quad \langle \langle \Phi(z) \Phi(\omega) \rangle \rangle = \frac{\gamma}{2(1 - \cos(z-\omega))}$$

$$= \theta^2 \left(\frac{2\gamma\pi}{\epsilon} - \pi^2 \gamma \right) = -\pi^2 \gamma \theta^2$$

$$\therefore \gamma = \frac{1}{\pi^2} \lambda \partial_\lambda \log \langle W_0 \rangle$$

Localization: $\langle W_0 \rangle = \frac{1}{N} L'_{N-1} \left(-\frac{\lambda}{4N} \right) e^{\frac{\lambda}{8N}}$, $\lambda = g_{\text{YM}}^2 N$
 $\lambda \partial_\lambda \langle W_0 \rangle = \frac{1}{2} \frac{\sqrt{\lambda} I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} + \mathcal{O}\left(\frac{1}{N^2}\right) \cong 2\pi^2 B$

② $\theta, \phi \ll 1$

$\theta = \phi \rightarrow$ BPS WL $\rightarrow \Gamma_{\text{cusp}} = 0$ if $\theta = \phi$

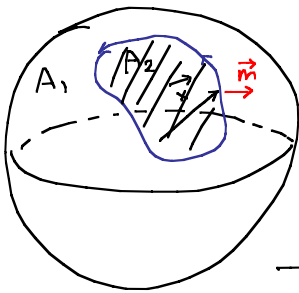
$\therefore \Gamma_{\text{cusp}} \cong (\theta^2 - \phi^2) B(\lambda)$

$\Rightarrow \theta = 0, \phi \ll 1$

$\Gamma_{\text{cusp}} \cong -\phi^2 B(\lambda)$

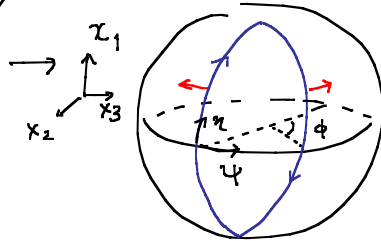
③ $(\theta = \phi) \frac{1}{8}$ BPS : $\Gamma_{\text{cusp}} = 0$

$S^2 \subset S^4, \vec{n} = (\vec{m}, 0, 0, 0)$ with $\vec{m} = \vec{x} \times \dot{\vec{x}}$



$\lambda \rightarrow \tilde{\lambda} = \lambda \frac{4A_1 A_2}{(A_1 + A_2)^2}$

$\langle W_\phi \rangle = \langle W_0 \rangle(\tilde{\lambda})$



$\tilde{\lambda} = \lambda \frac{4(\pi - \phi)(\pi + \phi)}{(2\pi)^2} = \lambda \left(1 - \frac{\phi^2}{\pi^2}\right)$

$\vec{x} = (\sin \eta, \cos \eta \cos \psi, \cos \eta \sin \psi)$

$\vec{m} = \dot{\vec{n}} = (0, \sin \psi, -\cos \psi)$

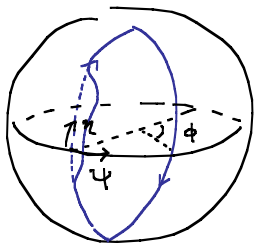
$= \begin{cases} (0, 0, -1) & \tau < 0 \\ (0, -\sin \phi, -\cos \phi) & \tau > 0 \end{cases} \rightarrow \theta = \phi$

$\sin \eta = \tanh \tau$

$\psi = \begin{cases} 0 & \tau < 0 \\ \pi - \phi & \tau > 0 \end{cases} \Rightarrow \dot{\psi} = 0$

$|\dot{\vec{n}}| = 1$

④ $|\theta - \phi| \ll 1$ ($\theta, \phi \sim \mathcal{O}(1)$)



slight deformation

displacement operator

$\delta W = \left[p \cdot \int dt (\delta x^j(t) D_j(t) + \delta \vec{n} \cdot \vec{\Theta}) \right] \cdot W$

$\vec{m}_{(\tau_0)} = \vec{m}_0 + \epsilon \left[(0, \sigma, 0) + \dot{\sigma} \left(\frac{1}{\cosh \tau}, -\tanh \tau, 0 \right) \right]$
 $(0, 0, -1) \quad \delta \phi = \epsilon \sigma, \quad \delta \theta = \epsilon (\sigma - \dot{\sigma} \tanh \tau)$

$\vec{m}_i \cdot \vec{m}_j \equiv \cos \theta = \cos \phi - \epsilon \sin \phi [\sigma - \dot{\sigma} \tanh \tau] \rightarrow \delta \theta = \epsilon (\sigma - \dot{\sigma} \tanh \tau)$

$\frac{\delta \langle W \rangle}{\langle W \rangle} = \int dt \left[\langle\langle D_i \rangle\rangle \delta x^i(t) + \langle\langle \vec{\Phi} \rangle\rangle \cdot \delta \vec{n} \right]$

$= \epsilon \int d\tau \left[\langle\langle D \rangle\rangle \sigma(\tau) + \langle\langle \vec{\Phi}^2 \rangle\rangle (\sigma(\tau) - \dot{\sigma} \tanh \tau) \right]$

τ -indep. (time transl.)

from $\frac{1}{8}$ BPS condition

$$\frac{\delta \langle W \rangle}{\langle W \rangle} \equiv \epsilon \partial_\phi \log \langle W_\phi \rangle = \epsilon \frac{\partial \tilde{\lambda}}{\partial \phi} \partial_{\tilde{\lambda}} \log \langle W_0 \rangle(\tilde{\lambda})$$

$$= -4\epsilon \frac{\phi}{(1 - \frac{\phi^2}{\pi^2})} B(\tilde{\lambda})$$

if $\dot{\sigma} = 0$, $\delta\phi = \delta\theta \rightarrow \delta \langle W \rangle = 0$ if $\langle\langle D \rangle\rangle + \langle\langle \bar{\Phi}^2 \rangle\rangle = 0$

$$\therefore \frac{\delta \langle W \rangle}{\langle W \rangle} = -\epsilon \langle\langle \bar{\Phi}^2 \rangle\rangle \int_{-\infty}^{\infty} dz \delta \text{th} z = \epsilon \langle\langle \bar{\Phi}^2 \rangle\rangle \underbrace{\int_{-\infty}^{\infty} dz \frac{\sigma(z)}{ch^2 z}}_{\approx \int dz \frac{1}{ch^2 z} = 2} = 2\epsilon \langle\langle \bar{\Phi}^2 \rangle\rangle$$

$$\Rightarrow \langle\langle D \rangle\rangle = -\langle\langle \bar{\Phi}^2 \rangle\rangle = \frac{2\phi}{1 - \frac{\phi^2}{\pi^2}} B(\tilde{\lambda})$$

$$\therefore \Gamma_{\text{cusp}} = -\langle\langle D \rangle\rangle \delta\phi - \langle\langle \bar{\Phi}^2 \rangle\rangle \delta\theta = (\delta\phi - \delta\theta) \langle\langle \bar{\Phi}^2 \rangle\rangle$$

$$= -(\phi - \theta) \frac{2\phi}{1 - \frac{\phi^2}{\pi^2}} B(\tilde{\lambda})$$

$$\rightarrow \Gamma_{\text{cusp}} = -\frac{(\phi^2 - \theta^2)}{(1 - \frac{\phi^2}{\pi^2})} B(\tilde{\lambda})$$

for any ϕ, θ , $|\phi - \theta| \ll \phi, \theta$

④ Any θ, ϕ [Integrability technique needed]

perturb. computation can be done upto λ^2 order

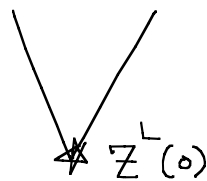
$$\Gamma_{\text{cusp}}(\lambda, \phi, \theta) = -\frac{\lambda}{8\pi^2} (\cos\theta - \cos\phi) \frac{\phi}{\sin\phi} + \left(\frac{\lambda}{8\pi^2}\right)^2 F(\theta, \phi) + \dots$$

} consistent with above for $\lambda \ll 1$

Nonpert. calculation for any θ, ϕ & λ

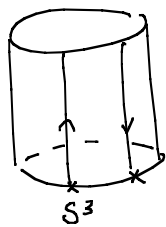
- Review of Integrability [Drukker; Correa, Maldacena, Sever]

- Insert Σ^L into WL.



$$\text{tr } \rho \left[e^{\int_0^\infty ds [iA_\mu \dot{x}^\mu + \vec{n} \cdot \vec{\Phi}(x)]} \Sigma^L_{(s=0)} e^{\int_{-\infty}^0 ds [\dots]} \right] : \text{non BPS } \rho \text{ except } \theta = \phi$$

Symmetry



$\mathbb{R} \times S^3 \rightarrow$ time dilation
 $SO(4) \rightarrow SO(3) \approx su(2)$ rotation around lines

$$\vec{n} = (0, 0, 0, 1, 0, 0) \quad \vec{n} \cdot \vec{\Phi} = \Phi^4$$

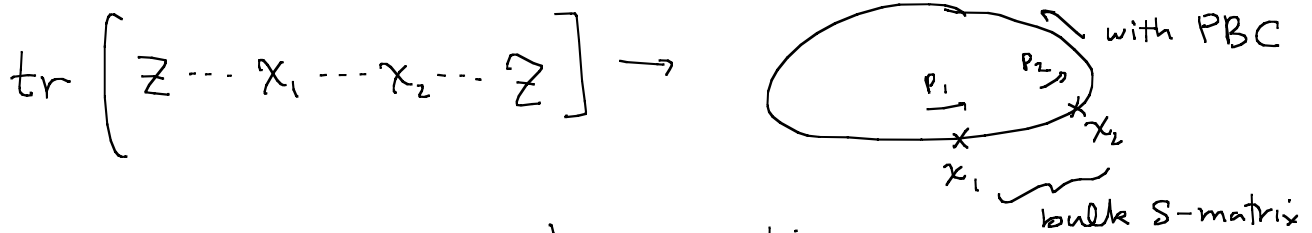
$SO(6) \rightarrow SO(5) \xrightarrow{\uparrow} SO(3) \approx su(2)$
 $Z = \Phi^5 + i\Phi^6$: sym which makes Z^L invariant

$$\Rightarrow su(2|2)$$

$$su(2|2) \otimes su(2|2) \xrightarrow{\text{breaks}} su(2|2)_D$$

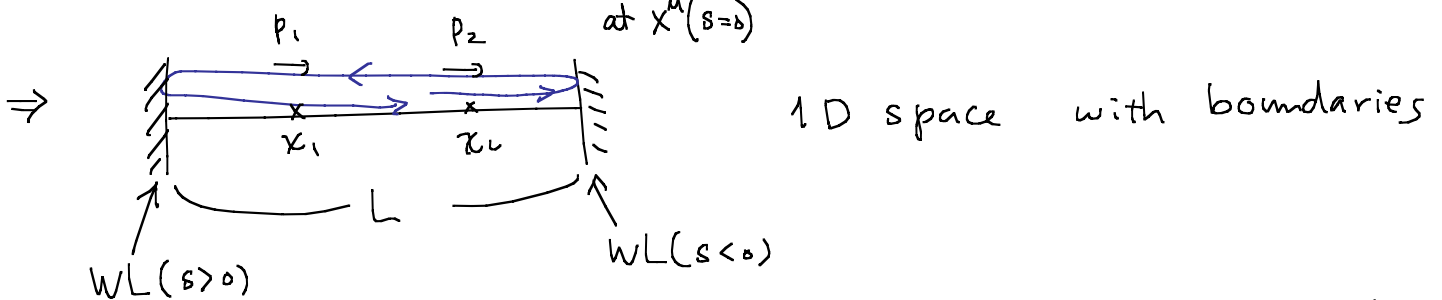
Operator space

for single trace operators; we mapped to (1+1) QFT



Wilson loop with one-operator insertion

$$\text{tr} \mathcal{P} \left[e^{\int_0^\infty ds [i A_\mu \dot{x}^\mu + \vec{n} \cdot \vec{\Phi} |\dot{x}|]} \underbrace{Z \dots \chi_i \dots Z}_{\text{at } x^\mu(s=0)} e^{\int_{-\infty}^0 ds [\dots]} \right] : \text{non}$$

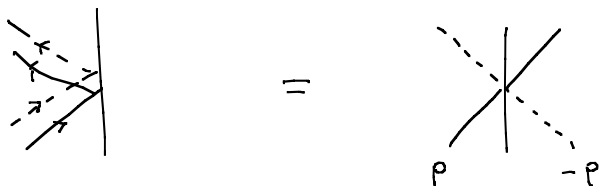


In addition to bulk-scattering, we need "boundary scattering"

$$R A_i(p) = R_i^j A_j(-p) \quad i, j \in su(2|2) \otimes su(2|2)$$

$$[R, Q_{su(2|2)_D}] = 0 \quad \Rightarrow \quad R_{(A \dot{A})}^{(B \dot{B})}(p) \propto S_{A \dot{A}}^{B \dot{B}}(p, -p)$$

$i = (A \dot{A})$
 $j = (B \dot{B})$

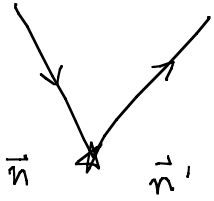


PBC \rightarrow Boundan BC.

$$e^{2iP_i L} [R_L(-p) S(p_i, p_i) \dots R_R(p) S(p_i, -p_i) \dots S(p_{N_i}, -p_i)] = 1$$

\swarrow SO(3) different SO(3)

With cusp $\vec{n} = (\underbrace{0, 0, 0, 1, 0, 0}$, $\vec{n}' = (\underbrace{0, 0, \sin\theta, \cos\theta, 0, 0}$)



$$X^\mu(s) = \begin{cases} (s, \underline{0, 0, 0}) & s \leq 0 \leftarrow \text{SO}(3) \\ (s \cos\phi, s \sin\phi, 0, 0) & s \geq 0 \leftarrow \text{different SO}(3) \end{cases}$$

can be described by rotation around (5-6 plane X^5)

$$m = \begin{pmatrix} e^{i\phi} & & & \\ & e^{-i\phi} & & \\ & & e^{i\phi} & \\ & & & e^{-i\phi} \end{pmatrix}$$

Lowts, IR
 $\downarrow \downarrow$
SU(2|2)

$$\underline{R_L = (m^{-1} R_R m)}$$

diagonalize \rightarrow new Asymp. BAE \rightarrow Boundary TBA, Y, Luscher.

if no bulk excitation; with only Z^L answer.

$$\Delta E \sim - \frac{\cos\phi - \cos\theta}{\sin\phi} \sum_{a=1}^{\infty} (-1)^a \left(\frac{-1 + \sqrt{1 + 16g^2/a^2}}{1 + \sqrt{1 + 16g^2/a^2}} \right)^{1+L} \frac{a \sin a\phi}{\sqrt{..}} F(a, g)$$

* weak coupling ($g \ll 1$) expansion ($\& L \rightarrow \infty$)

$$\Delta E \rightarrow \Gamma_{\text{cusp}} = -4g^2 \frac{\cos\phi - \cos\theta}{\sin\phi} \sum_{a=1}^{\infty} (-1)^a \frac{\sin a\phi}{a} = 2g^2 \frac{\cos\phi - \cos\theta}{\sin\phi} \phi \checkmark$$

Now, I am interested in ABJM.

Supersymmetric WLS

1/6 BPS [Drukker-Plefka-Yang; Chen-Wu; Rey-Suyama-Yamaguchi.]

★ 1/2 BPS WL [Drukker-Trancanelli]

WL

$$[\hat{A}_\mu] \leftarrow M \times M$$

$$[A_\mu] = [\Psi] = 1, \quad [C^I] = \frac{1}{2} \underbrace{U(N)_k \times U(M)_{-k}}_{\text{bifundamental}}$$

$\xleftrightarrow{\text{bifundamental}}$

Try

$$L = A_\mu \dot{x}^\mu + i\dot{x} | M_J^I C_I \bar{C}^J + i\dot{x} | \cancel{\Psi}$$

$$\therefore L = \begin{pmatrix} \xleftarrow{N} & \xrightarrow{M} \\ \begin{matrix} \uparrow N \\ \downarrow M \end{matrix} & \begin{matrix} \xrightarrow{N} & \xrightarrow{M} \\ \begin{matrix} \uparrow N \\ \downarrow M \end{matrix} & \begin{matrix} \uparrow N \\ \downarrow M \end{matrix} \end{matrix} \left(\begin{array}{c|c} A_\mu \dot{x}^\mu + \frac{2\pi}{k} i\dot{x} | M_J^I C_I \bar{C}^J & \sqrt{\frac{2\pi}{k}} i\dot{x} | \eta_I^\alpha \bar{\Psi}_\alpha^I \\ \hline \sqrt{\frac{2\pi}{k}} i\dot{x} | \Psi_I^\alpha \bar{\eta}_\alpha^I & \hat{A}_\mu \dot{x}^\mu + \frac{2\pi}{k} i\dot{x} | \hat{M}_J^I \bar{C}^J C_I \end{array} \right)$$

$$W_R \equiv \text{Tr}_R \mathcal{P} \exp \left(i \oint_C L d\tau \right)$$

\uparrow
U(N|M) rep.

$$A_\mu \dot{x}^\mu = A_0 \quad \text{let } A_0 \equiv A_0 + \frac{2\pi}{k} M_J^I C_I \bar{C}^J$$

$$\hat{A}_0 \equiv \hat{A}_0 + \frac{2\pi}{k} \hat{M}_J^I \bar{C}^J C_I$$

$$\text{if } M_J^I = \hat{M}_J^I = \begin{pmatrix} - & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \eta_1^+, \bar{\eta}_+^1 \neq 0$$

Insert Bulk BPS operator.

$$\text{Tr}_R \mathcal{P} \left[e^{i \int_0^\infty L d\tau} \left[(C, \bar{C}^1) \dots (C, \bar{C}^1) \right] (0) e^{i \int_\infty^0 L d\tau} \right]$$

{ without cusp; "1/6 BPS" ✓

{ with cusp; How much is known by "Localization"?

- Integrability of ABJM

- symmetry of boundary s-matrix