Dimer integrable systems

Haniltonians on a chess board

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Motivation

Dimer model, also known as brane tiling or domino tiling, is a study of tessellation of Euclidean plane by dominoes.
Equivalently, it is a study of perfect matching on a lattice graph.

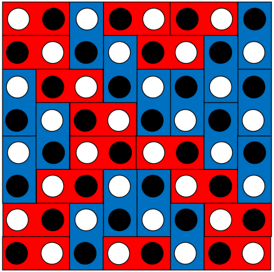

Figure. 1: A domino tiling of 8×8 chess board.

Motivation

Why are physicists interested in dimer model?

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- There is a one-to-one correspondence between periodic dimer model and a ground state of fully-frustrated Ising model on 2d periodic lattice. [Barahona '82]
- Any dimer model on a torus defines a relativistic integrable system whose conserving Hamiltonians can be read off from the *loops* of the dimer graph. [Goncharov, Kenyon '11]

Dimer model

A bipartite graph $\Gamma = (B, W, E)$ is a triad embedded on an oriented 2-surface $\mathcal{S}$:

- a finite set of black nodes B,
- a finite set of white nodes W, and
- a finite set E of edges, consisting of embedded closed intervals e on $\mathcal{S}$ such that one boundary of e belongs to B and the other boundary belongs to W.

such that any edge can intersect another edge only at its boundary.
$\Gamma = (B, W, E)$ is a dimer model if

- Every equivalent node is on the boundary of $\mathcal{S}$, and
- every faces of Γ is simply-connected.

Dimer model

Here we only consider the case $\mathcal{S} = T^2$. The periodicity on the torus is realized by the *unit cell*.

A perfect matching $M \in E$ is a collection of edges such that all nodes in a unit cell is connected by exactly one edge.

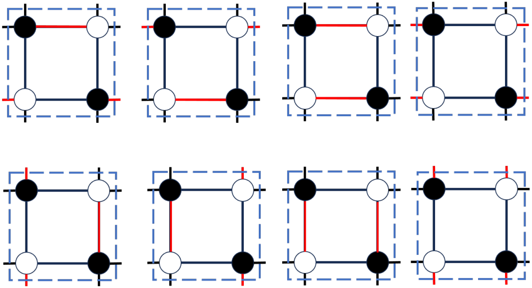
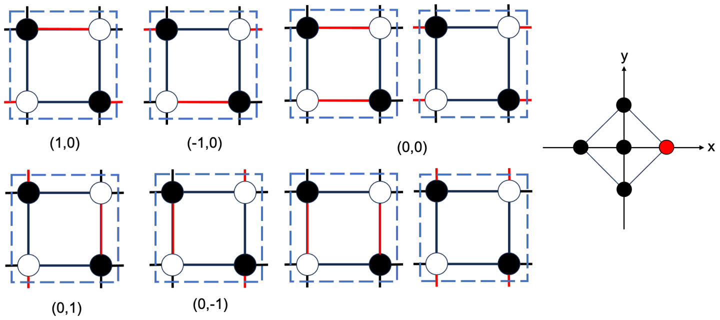


Figure. 2: The eight perfect matching of a 2×2 square dimer graph.

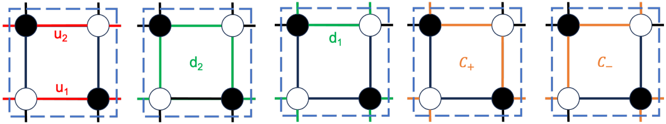
Dimer model

We choose the default orientation of the edges to be pointing from a black node to a white node.
A weight is assigned to a perfect matching M based on the orientation of its edges that pass through the boundary of the unit cell.



Dimer model

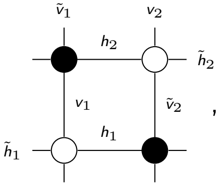
The 1-loops are paths on the dimer graph connecting from one white node to the same white node across unit cell boundary. The 1-loops are obtained by taking difference between a perfect matching with the reference perfect matching. There are four 1-loops and two Casimir $C_{\pm}$ in the 2×2 square dimer:



The n-loop is the product of n non-overlapping 1-loops. There is a single 2-loop $u_1 u_2$ for the 2×2 square dimer.

Dimer model

The spectral curve is obtained by Kasteleyn matrix [Kenyon '03]:



$$, K = \begin{pmatrix} h_1 - \tilde{h}_1 X & v_2 Y + \tilde{v}_2 \\ v_1 + \frac{\tilde{v}_1}{Y} & -h_2 + \frac{\tilde{h}_2}{X} \end{pmatrix}.$$

The spectral curve:

$$\Sigma = \left\{ (X, Y) \in \mathbb{C}^2 | \; 0 = \det K = \tilde{h}_1 h_2 \left[X - H_1 + \frac{H_2}{X} - C_+ Y - \frac{C_-}{Y} \right] \right\}$$

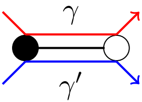
with $H_1 = u_1 + u_2 + d_1 + d_2, \;\; H_2 = u_1 u_2.$

Dimer integrable system

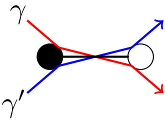
Define Poisson commutation relation between two 1-loops sharing an edge [Goncharov, Kenyon '03]:

$$\{\gamma, \gamma'\} = \epsilon_{\gamma,\gamma'}\gamma\gamma', \;\; \epsilon_{\gamma,\gamma'} = \sum_{v} \mathsf{sgn}(v)\delta_v(\gamma, \gamma')$$

$\delta_v \in \frac{1}{2}\mathbb{Z}$ is a skew-symmetric bilinear form. Examples that we will encounter:



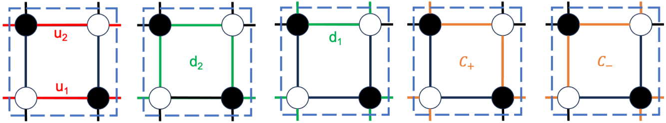
$$\{\gamma, \gamma'\} = \gamma\gamma'.$$



$$\{\gamma, \gamma'\} = 0.$$

Dimer integrable system

The four 1-loops $u_n, \; d_n, \; n = 1, 2$, in the 2×2 square dimer



satisfy

$$\{u_n, d_n\} = u_n d_n, \; \{d_2, u_1\} = d_2 u_1, \; \{d_1, u_2\} = d_1 u_2. \qquad (1)$$

The Casimirs $C_{\pm}$ commute with all 1-loops.

Dimer integrable system

The 1-loops in 2×2 square dimer can be expressed by canonical coordinates (q_n, p_n), $p_{n+2} = p_n$, $q_{n+2} = q_n$:

$$u_n = e^{p_n}, \; d_n = R^2 e^{q_{n-1} - q_n} e^{p_n} \tag{2}$$

The commuting Hamiltonians are

$$\begin{aligned} H_1 &= \sum_{n=1}^{2} u_n + d_n = \sum_{n=1}^{2} e^{p_n} + R^2 e^{q_{n-1} - q_n} e^{p_n} \\ H_2 &= u_1 u_2 = e^{p_1 + p_2}. \end{aligned} \tag{3}$$

It is the type A relativistic Toda lattice of two particles.

Dimer integrable system

Definition

A n-loop is a product of n non-overlapping 1-loops.

Definition

The n-th Hamiltonian H_n is a sum over n-loops:

$$H_n = \sum n\text{-loops}.$$

Theorem [Goncharov-Kenyon]

A dimer graph defines an integrable system.

$$\{H_n, H_m\} = 0, \;\; n, m = 1, \ldots, N$$

Dimer integrable system

Relativistic Toda lattice (RTL) belongs to a family of cluster integrable system called $Y^{N,k}$ dimer model with spectral curve:

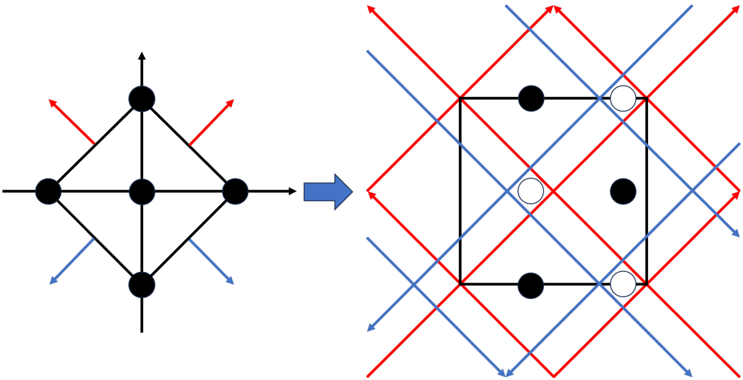
$$\Sigma = \left\{ (X,Y) \in \mathbb{C}^2 | C_+ Y + \frac{C_- X^k}{Y} = T(X) \right\}, \ T(X) = \sum_{n=0}^{N} H_n X^{n-\frac{N}{2}}.$$

The dual graphs of these dimer graphs are planar, periodic quiver gauge theories which arise from a stack of D3 branes probing a singular, toric CY3. [Franco, Hanany, Kennaway, Vegh, Wecht '05]
A string dual exists on $AdS_5 \times X_5$, where X_5 is a Sasaki-Einstein manifold whose metric $Y^{N,k}$ is labeled by two integers $k \leq N$.

[Benvenuti, Hanany, Kazakopoulos '04].

Dimer integrable system

A dimer graph can be construct based on a given toric diagram:



$Y^{N,0}$ square dimer

The 1-loops $\{u_n, d_n\}_{n=1}^{N}$ obeys

$$\{u_n, d_n\} = u_n d_n,$$
$$\{d_{n+1}, u_n\} = d_{n+1} u_n,$$
$$\{d_{n+1}, d_n\} = d_{n+1} d_n.$$

The first Hamiltonian H_1 is $\hat{A}_{N-1}$ relativistic Toda Hamiltonian.

$$H_1 = \sum_{n=1}^{N} \left(1 + R^2 e^{\mathrm{q}_n - \mathrm{q}_{n-1}}\right) e^{-\mathrm{p}_n}.$$

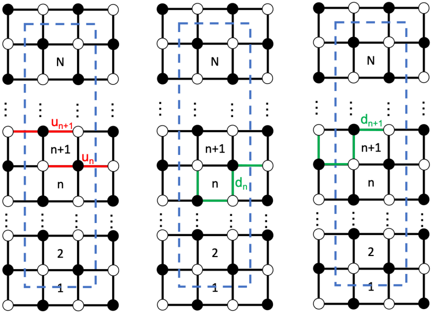


Figure. 3: The 1-loops in a $Y^{N,0}$ square dimer model when N is even.

Hexagon $Y^{N,N}$ dimer

$Y^{N,N}$ model is the hexagon diagram.
The 1-loops $\{u_n, d_n\}_{n=1}^{N}$ obeys

$$\{u_n, d_n\} = u_n d_n,$$
$$\{d_{n+1}, u_n\} = d_{n+1} u_n,$$

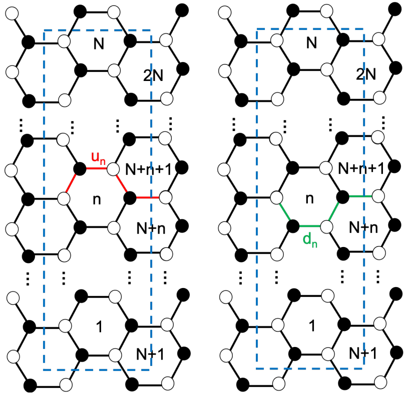


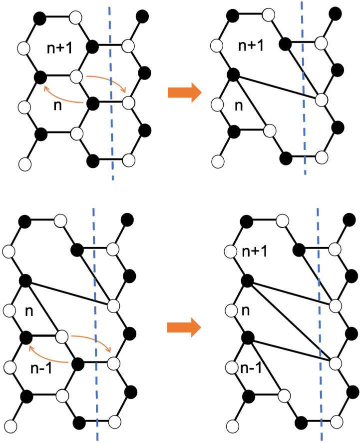
Figure. 4: The 1-loops in a $Y^{N,N}$ hexagon dimer model.

$Y^{N,k}$ dimer

Eager, Franco, and Schaeffer prove that $Y^{N,k}$ dimer graph can be obtained by gluing vertexes in $Y^{N,N}$ model consecutively at $k, \ldots, N-1$ hexagons. [Eager, Franco, Schaeffer '11]

The non-vanishing 1-loop Poisson commutations are

$$\{u_n, d_n\} = u_n d_n,$$
$$\{d_{n+1}, u_n\} = d_{n+1} u_n,$$
$$\{d_{n+1}, d_n\} = d_{n+1} d_n, \quad n = k, \ldots, N-1.$$



Remark: All first Hamiltonian of $Y^{N,k}$ model has the same non-relativistic limit $p_n \to Rp_n$, $R \to 0$:

$$\lim_{R\to 0} H_1|_{Y^{N,k}} = N + R\sum_{n=1}^{N} p_n + R^2\left[\sum_{n=1}^{N} \frac{p_n^2}{2} + e^{q_n - q_{n-1}}\right] + \mathcal{O}(R^3)$$

The R^2 term is the non-relativistic $\hat{A}_{N-1}$ Toda lattice.

New dimer models

Here we introduce two ways to modify existing dimer graphs to generate new ones:

- Non-standard gluing [NL '23]
- Introducing impurity [Lee-NL '24]

Non-standard gluing

Denote $S \subset \{1, 2, \ldots, N\}$ as the gluing set with $|S| = N - k$. $Y^{N,k}[S]$ is obtaining by gluing at the n-th hexagon if $n \in S$.

$$G_S(n) = \begin{cases} 0 & n \in S \\ 1 & n \notin S \end{cases}.$$

The non-vanishing 1-loop Poisson commutation relations are

$$\{u_n, d_n\} = u_n d_n,$$
$$\{d_{n+1}, u_n\} = d_{n+1} u_n,$$
$$\{d_{n+1}, d_n\} = d_{n+1} d_n \text{ if } n \in S.$$

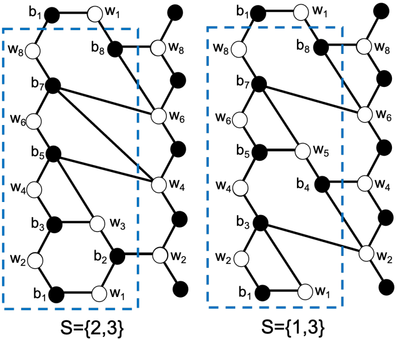


Figure. 5: Two non-equivalent gluing of $Y^{4,2}[S]$ dimer. The standard dimer graph on the left and the alternative on the right.

Non-standard gluing

The third Hamiltonian of the standard $Y^{4,2}[\{2,3\}]$ dimer:

$$
\begin{aligned}
H_3|_{\text{standard}} = &\, u_3 u_2 u_1 + u_2 u_1 u_4 + u_1 u_4 u_3 + u_4 u_3 u_2 \\
&+ u_3 u_2 d_1 + u_2 u_1 d_4 + u_1 u_4 d_3 + u_4 u_3 d_2 \\
&+ u_3 d_2 d_1 + u_2 d_1 d_4 + d_2 d_1 d_4
\end{aligned} \tag{4}
$$

The third Hamiltonian of the non-standard $Y^{4,2}[\{1,3\}]$ dimer:

$$
\begin{aligned}
H_3|_{\text{non-standard}} = &\, u_3 u_2 u_1 + u_2 u_1 u_4 + u_1 u_4 u_3 + u_4 u_3 u_2 \\
&+ u_3 u_2 d_1 + u_2 u_1 d_4 + u_1 u_4 d_3 + u_4 u_3 d_2 \\
&+ u_2 d_1 d_4 + u_4 d_3 d_2
\end{aligned} \tag{5}
$$

Non-standard gluing

Question: What is the relation between $Y^{N,k}[S]$ model with different gluing sets S?

Non-standard gluing

Question: What is the relation between $Y^{N,k}[S]$ model with different gluing sets S?
Conjecture: Seiberg duality of $Y^{N,k}$ quiver gauge theories?

Introduce impurities

Our aim here is to find dimer graphs for RTL of type B,C,D. Sklyanin proved that type B,C,D RTL can be viewed as type A with special boundary conditions. Spectral curves are known through the Lax formalism. [Sklyanin '88]

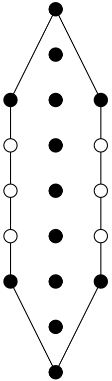
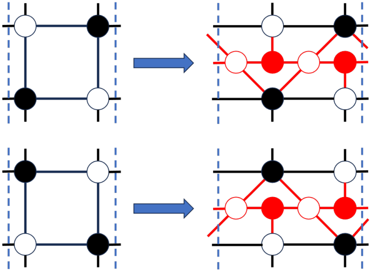


Figure. 6: Toric diagram associated to spectral curve of $\mathfrak{so}(8) = D_4$ RTL.

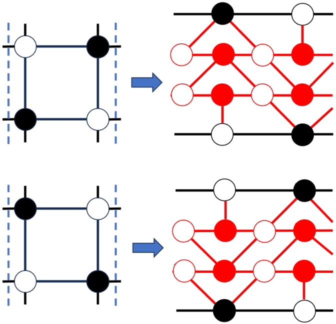
Introducing Impurities

The dimer graph for D_N can be build based on $Y^{2N-4,0}$ square dimer. A pair of vertical lines in the toric diagram introduces impurity to the square dimer graph. The newly introduced red nodes modifies the edges inside the square.



Introducing Impurities

It is allowed to introduce impurity multiple times in a single square in the Y^{2N-4} dimer.
Two pairs of vertical lines in the toric diagram introduces double impurity to the square dimer graph. The newly introduced red nodes modifies the edges inside the square.



Introducing Impurity

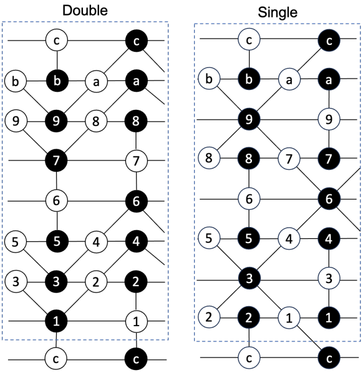
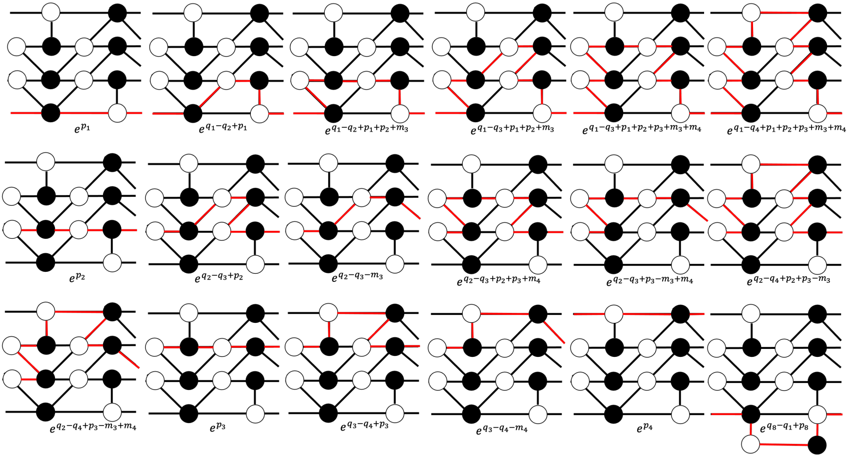


Figure. 7: Two bipartite graphs generated by toric diagram for D_4. The left graph is constructed by placing double impurity at the first and third square in $Y^{4,0}$ graph. The right one place single impurity in each of the square in the $Y^{4,0}$ graph.

Introducing Impurity



Introducing Impurity

To obtain dimer for $\hat{D}_N$ RTL, we perform folding through

- Two double impurities placed furthest away in $Y^{2N-4,0}$ dimer.
- Assign $2N+2N$ canonical coordinates to the 1-loops based on their Poisson commutation.
- Cut $\#$ canonical coordinates by half by requiring $H_n = H_{2N-n}$.
- Canonical transformation at two double impurities:

$$e^q \to \frac{\cosh \frac{p}{2}}{\sinh q}, \quad e^p \to \frac{\cosh \frac{p-2q}{2}}{\cosh \frac{p+2q}{2}}.$$

$H_1 = H_{2N-1}$ recovers $\hat{D}_N$ Toda lattice Hamiltonian.

Quantization

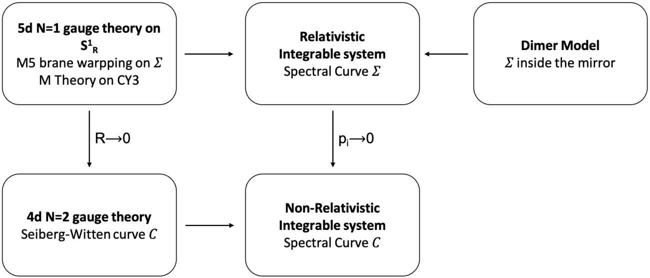
The classical system can be promote to quantum by replacing

$$\{\gamma, \gamma'\} = \gamma\gamma' \to \hat{\gamma}\hat{\gamma}' - e^{-\hbar}\hat{\gamma}'\hat{\gamma} = 0.$$

Question: How to solve for wavefunction?

Quantization

Bethe/Gauge correspondence:



Promotion to quantum with Ω-deform in 4d/5d gauge theory. The stationary states of the quantum integrable system are the vacua of the effective 2d $\mathcal{N} = (2,2)$ or 3d $\mathcal{N} = 2$ theories.

Quantization

Couple 1d fermionic d.o.f to bulk 5d theory s.t. it is half-BPS.

$$S_{1d} = \int_{S^1} dt \; \chi^\dagger(\partial_t - iA_t + \Phi + x)\chi.$$

It is a co-dim 4 observable in 5d theory. By localization

$$\frac{\mathcal{Z}_{1d/5d}}{\mathcal{Z}_{5d}} = \langle \mathcal{X}_k(X = e^x) \rangle = \left\langle Y(Xe^{\varepsilon_1+\varepsilon_2}) + \frac{\mathfrak{q}X^k}{Y(X)} \right\rangle = \sum_{n=0}^{N} X^{n-\frac{N}{2}} W_{\wedge^n}$$

$W_{\wedge^n}$: n-th anti-sym. rep. of Wilson loop. [Tong, Wong '14] [Kim '16]
$\mathcal{X}_k(X)$: *qq-character*, whose vev is regular in x. It is the quantum uplift of the Seiberg-Witten curve. [Nekrasov '15]

Quantization

Gauge theory placed on orbifolded spacetime $\hat{\mathbb{C}}_1 \times \left(\hat{\mathbb{C}}_2/\mathbb{Z}_N\right)$. The quotient space can be identified with $\mathbb{C}^2_{12}$ the complex manifold via

$$\begin{aligned} \hat{\mathbb{C}}_1 \times \left(\hat{\mathbb{C}}_2/\mathbb{Z}_N\right) &\to \mathbb{C}^2_{12} \\ (\hat{\mathbf{z}}_1, \hat{\mathbf{z}}_2) &\mapsto \left(\mathbf{z}_1 = \hat{\mathbf{z}}_1, \mathbf{z}_2 = \hat{\mathbf{z}}_2^N\right) \end{aligned} \tag{6}$$

The theory on the orbifold space $\hat{\mathbb{C}}_1 \times \left(\hat{\mathbb{C}}_2/\mathbb{Z}_N\right)$ is equivalent to gauge theory on the smooth space $\mathbb{C}^2_{12}$ with a specific boundary condition along $\mathbb{C}_1$ on $\hat{\mathbf{z}}_2 = 0$

$$A_\mu dx^\mu \sim \mathrm{diag}(\alpha_1, \ldots, \alpha_N) d\theta. \tag{7}$$

Gauge symmetry is broken to its maximal torus $U(1)^N \subset U(N)$.

Monodromy defect

By localization, the path integral of the 5d $\mathcal{N}=1$ gauge theory on a orbifold reduces to a finite dimensional integral over the instanton moduli space $\mathfrak{M}^{\text{orb}}_{\hat{\mathbb{C}}^2_{12}}$.

It can be constructed by ADHM construction described by the $\mathbb{Z}_N$ chainsaw quiver. [Kanno, Tachikawa '11]

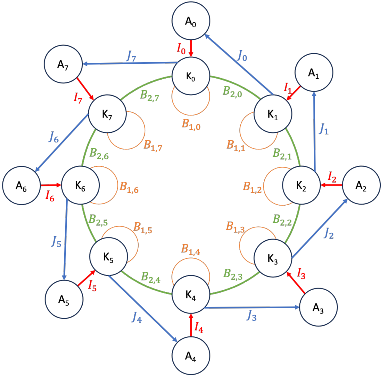


Figure. 8: $\mathbb{Z}_8$ Chainsaw quiver

Quantization

There exists a natural projection $\rho : \mathfrak{M}^{\mathrm{orb}}_{\hat{\mathbb{C}}^2_{12}} \to \mathfrak{M}_{\mathbb{C}^2_{12}}$.

- Integration over $\mathfrak{M}^{\mathrm{orb}}_{\hat{\mathbb{C}}^2_{12}} \implies$ Integration over $\mathfrak{M}_{\mathbb{C}^2_{12}}$ + integration over fiber of the projection.

$$\hat{\mathcal{Z}}_{\hat{\mathbb{C}}^2_{12}} = \int_{\mathfrak{M}^{\mathrm{orb}}_{\hat{\mathbb{C}}^2_{12}}} 1 = \int_{\mathfrak{M}_{\mathbb{C}^2_{12}}} \Psi(\hat{\mathfrak{q}}) = \langle \Psi(\hat{\mathfrak{q}}) \rangle \mathcal{Z}_{\mathbb{C}^2_{12}} \tag{8}$$

$\hat{\mathfrak{q}} = (\mathfrak{q}_\omega)_{\omega=0}^{N-1}$ are counting parameters of $\mathbb{Z}_N$-orbifold charges. The defect is characteried by bijective coloring function of Coulomb moduli

$$c : \{1, \ldots, N\} \to \mathbb{Z}_N$$

and fractional CS-levels on the 3d theory k_n obeying

$$k_n \in \{0,1\}, \ \sum_{n=1}^{N} k_n = k.$$

Quantization

The line defect on $S^1 \times \hat{\mathbb{C}}^2_{12}$ as fractional *qq*-character is a regular function in x:

$$\begin{aligned}\langle \mathcal{X}_\omega(X = e^x) \rangle_{\mathbb{Z}_N} &= \frac{1}{\hat{\mathcal{Z}}_{\hat{\mathbb{C}}^2_{12}}} \sum_{\hat{\lambda}} \prod_{\omega=0}^{N-1} \mathfrak{q}_\omega^{k_\omega} \hat{\mathcal{Z}}[\hat{\lambda}] \mathcal{X}_\omega(x)[\hat{\lambda}] \\ &= \langle Y_{\omega+1}(X e^{\varepsilon_1}) \rangle_{\mathbb{Z}_N} + \left\langle \frac{\mathfrak{q}_\omega X^{k_\omega}}{Y_\omega(X)} \right\rangle_{\mathbb{Z}_N} .\end{aligned}$$

Obtain Hamiltonian

A function $f(X=e^x)$ regular in x means it can only has pole at $X=\infty$ and $X=0$. Consider $X>>1$ and $X<<1$ expansion of $\langle \mathcal{X}_\omega(X)\rangle/\sqrt{X}$:

- Large X:

$$\frac{\langle \mathcal{X}_\omega(X)\rangle_{\mathbb{Z}_N}}{\sqrt{X}} = \sum_{j=0}^{\infty} \langle c_{j,\omega}^{(+)}\rangle_{\mathbb{Z}_N} X^{-j}$$

- Small X:

$$\frac{\langle \mathcal{X}_\omega(X)\rangle_{\mathbb{Z}_N}}{\sqrt{X}} = \frac{1}{X}\operatorname*{Res}_{X=0}\frac{\langle \mathcal{X}_\omega(X)\rangle_{\mathbb{Z}_N}}{\sqrt{X}} + \sum_{j=0}^{\infty} \langle c_{j,\omega}^{(-)}\rangle_{\mathbb{Z}_N} X^{j}$$

Taking difference and matching X^{-1} coefficient gives

$$\left\langle c_{1,\omega}^{(+)} - \operatorname*{Res}_{X=0}\frac{\mathcal{X}_\omega(X)}{\sqrt{X}}\right\rangle_{\mathbb{Z}_N} = 0.$$

Obtain Hamiltonian

Take linear combination

$$\sum_{\omega=0}^{N-1} \hat{C}_\omega \left\langle c_{1,\omega}^{(+)} - \operatorname*{Res}_{X=0} \frac{\mathcal{X}_\omega(X)}{\sqrt{X}} \right\rangle_{\mathbb{Z}_N} \hat{\mathcal{Z}}_{\hat{\mathbb{C}}^2_{12}} = 0$$

with proper coefficients $\hat{C}_\omega$. We obtain

$$\hat{\mathrm{H}}_{N-1}|_{Y^{N,k}[S]} \langle \Psi(\hat{\mathfrak{q}}) \rangle = \langle \mathbf{S}\Psi(\hat{\mathfrak{q}}) \rangle,$$

with $\mathfrak{q}_\omega = R^2 e^{q_{\omega+1}-q_\omega}$ and

$$G_S(n) = k_n.$$

Obtain Hamiltonian

In the NS-limit the bulk instanton is locked to limit shape $\mathbf{\Lambda}$.

$$\lim_{\varepsilon_2 \to 0} \langle \Psi(\hat{\mathrm{q}}) \rangle = \psi(\hat{\mathrm{q}}), \quad \lim_{\varepsilon_2 \to 0} \langle \mathbf{S}\Psi(\hat{\mathrm{q}}) \rangle = \mathbf{S}[\mathbf{\Lambda}]\psi(\hat{\mathrm{q}})$$

We obtain Schrödinger equation:

$$\hat{\mathrm{H}}_{N-1}|_{Y^{N,k}[S]}\psi(\hat{\mathrm{q}}) = \mathbf{S}[\mathbf{\Lambda}]\psi(\hat{\mathrm{q}}).$$

Remark: $\psi(\hat{\mathfrak{q}})$ is the common eigenfunction of all commuting Hamiltonians. This is proven by constructing the Lax matrices (and reflection matrices in the case of type D) of the integrable system from *qq*-character.

Summary:

- We construct new dimer integrable systems by modifying existing dimer graph.
- The wavefunction of quantum dimer integrable system is proven to be co-dimensional two monodromy defect in 5d $\mathcal{N} = 1$ gauge theory.

Future direction:

- Find dimer graph for type B, C or even E.
- The BPS quiver dual to the modified dimer graphs.
- More general modification of dimer graph.
- Dimer graphs for non-toric SUSY theories or non-convex toric diagrams.

Thank you for your attention!