APCTP Focus Program "Integrability, Duality, and Related Topics" 2024

Black Holes from Young Diagrams

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This talk is based on 2409.15751 with

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and

the ongoing works with

Robert de Mello Koch (Huzhou university),
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Jehyun Lee (SNU).

Motivation and Introduction

Motivation 1

- Quantum gravity is not just an intellectual desire to describe all interactions as one, but is absolutely necessary to explain real physical situations such as early universe or black holes.

- In particular, many theoretical properties of black holes remain ununderstood, despite the fact that their reality has become clear from recent observational astronomy and gravitational wave experiments.

- What are black holes? What characterestics do define black holes? These are not easy to answer.

- Even when it comes to BPS black holes, which are theoretically much simpler in the sense that they have greater symmetry, there is still much we don't know.

- For example, what are the holographic dual operators of black holes?

Motivation 2

- In AdS/CFT context, there were some notable breakthroughs in the holographic dual of black holes last year and the year before last. [Chang, Lin][Choi, Kim,...]

- However, those constructions were only succesful in low ranks of gauge groups such as SU(2) and SU(3) and performed through a brute force method.

- Can we construct dual operators in generic U(N) gauge group which allows large N limit?

- Can we think of a systematic method that can overcome the technical difficulties of the approach that forced us to use the brute force method?

- The story I want to talk about today is about these questions, and while it cannot give you complete answers because it is still a work in progress, I want to at least talk about where the promising direction is.

Motivation 3

- The study of black holes through holography already has a long history. In particular, it is diverse, including methods through thermal field theory, matrix models containing the characteristics of black holes, gravitational path integrals, index calculations using the BPS property, and recent cohomology problems.

- I do not have the ability to introduce them in detail, so I would like to list some interesting characteristics.

- The AdS BHs have some interesting features such as

  (1) # of states in microcanonical ensemble is order of e^{N^2}.
  (2) There are small AdS BH and large AdS BH which are qualitatively different from each other.
  (3) Energy gap between BPS BHs and non-BPS BHs [Turiaci et al.]
  (4) Fast scrambling and chaos (e.g. Maldacena, Shenker, Stanford bound)

- It would be great if we can explain these features in dual gauge theory, N=4 supersymmetric Yang-Mills theory.

Planar limit is not working even in large N

- In planar limit of AdS/CFT, we have integrability in the spectral problem. This allows us a very powerful exact results. We can explore finite coupling results and discuss nonperturbatively dynamical scatterings. However, this planar sector cannot capture black holes.

- By MSS, a black hole is shown to be a maximally chaotic physical object. Namely it is far from integrability, which can be said to be the opposite of chaos. Also, since the planar limit is related to stringy dynamics, it is generally difficult to apply it to the cases such as D-branes or black holes.

- To figure this out, note that the planar limit must be distinguished from the large N limit. To be precise, in addition to being large N, the size of the operator being considered is only allowed up to stringy size ~ O(\sqrt{N}).

- Reason for that is

$$\langle \mathcal{O}_J \mathcal{O}_J^\dagger \rangle = J N^J + J^5 N^{J-2} + \dots$$

- So, if J ~ \sqrt{N}, the second term(nonplanar) is comparable to the first term. So, beyond this order, nonplanar diagrams cannot be neglected. We must simply sum everything.

- It is also noticable that nonplanar corrections make a transition from integrability to chaos. ['20 Mcloughlin et al.]

Heavy operator with O(N^2) dimension 1

- Thus, to describe BHs in gauge theory side, we have to consider huge operators.

J	object
$O(1)$	graviton
$O(\sqrt{N})$	string
$O(N)$	giant graviton
$O(N^2)$	new spacetime geometry

- The holographic dual of BHs must have bare conformal dimensions of order N^2.
- This requirement is also related to the fact that we need strong gravity backreaction. O(N^2) objects completely change geometry.
- So, black holes are highly nonplanar. In some sense, we may say that the ``planar'' AdS5 x S5 is a kind of vacuum since it is stringy, and BHs are excited states.
- How can we construct such a heavy operator?

Heavy operator with O(N^2) dimension 2

- The easiest way is to use multi-trace operator basis.
- A tedious problem is to impose trace relations in a given N. For example, for n=3 and N=2, we have

$$\mathrm{Tr}\left(Z^3\right) = \frac{1}{2}\left[3\mathrm{Tr}\left(Z^2\right)\mathrm{Tr}\left(Z\right) - \left(\mathrm{Tr}\left(Z\right)\right)^3\right]$$

- This is surely possible but not systematic.
- There exists a complete but not overcomplete basis for the local gauge invariant operators called Schur polynomials. Here, the operators are constructed using group representation theory.

$$\chi_R(U) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)U^{i_1}_{i_{\sigma(1)}}U^{i_2}_{i_{\sigma(2)}}\cdots U^{i_n}_{i_{\sigma(n)}}$$

- Schurs diagonalize the free field theory two point function and they take finite N constraints into account and they can often be used to compute higher point functions.

Heavy operator with O(N^2) dimension 3

- With Schur basis, it is quite easy to construct operators which scale as O(N^2). Simply, take O(N^2) number of boxes in Young diagrams.
- A nice example is the 1/2 BPS sector. The Schur polynomials with only Zs provide a complete basis of the sector. The Young diagrams show a very immediate connection to free fermions.
- The free fermion phase space also makes a visible appearance in the corresponding supergravity geometries, known as the LLM geometries and it is possible to give a very concrete link between conformal field theory operators and LLM geometries ['04 Lin, Lunin, Maldacena].
- As in the 1/2-BPS geometry, if the Schur operator is useful in describing a new spacetime holographically, can a black hole geometry be also described through Schur polynomials?
- What should we further consider? Can we check such a construction is correct?

Schur polynomials 1

- Schur polynomial

$$\chi_R(U) \quad = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) U^{i_1}_{i_{\sigma(1)}} U^{i_2}_{i_{\sigma(2)}} \cdots U^{i_n}_{i_{\sigma(n)}}$$

- By manipulating symmetric group S_n, we write a specific linear combination of multi-trace operators with size n. The “R” means a representation of S_n and is completely classified in terms of Young diagrams. The coefficients are characters of relevant permutation groups. For example,

$$\begin{aligned}\chi_{\square\square\square}(Z) &= \frac{1}{3!}(\chi_{\square\square\square}(\mathbb{1}) Z^{i_1}_{i_i} Z^{i_2}_{i_2} Z^{i_3}_{i_3} + \chi_{\square\square\square}((1\ 2)) Z^{i_1}_{i_2} Z^{i_2}_{i_1} Z^{i_3}_{i_3} + \chi_{\square\square\square}((1\ 3)) Z^{i_1}_{i_3} Z^{i_2}_{i_2} Z^{i_3}_{i_1} \\ &\quad + \chi_{\square\square\square}((2\ 3)) Z^{i_1}_{i_1} Z^{i_2}_{i_3} Z^{i_3}_{i_2} + \chi_{\square\square\square}((1\ 2\ 3)) Z^{i_1}_{i_2} Z^{i_2}_{i_3} Z^{i_3}_{i_1} + \chi_{\square\square\square}((1\ 3\ 2)) Z^{i_1}_{i_3} Z^{i_2}_{i_1} Z^{i_3}_{i_2}) \\ &= \frac{1}{6}(Z^{i_1}_{i_i} Z^{i_2}_{i_2} Z^{i_3}_{i_3} + Z^{i_1}_{i_2} Z^{i_2}_{i_1} Z^{i_3}_{i_3} + Z^{i_1}_{i_3} Z^{i_2}_{i_2} Z^{i_3}_{i_1} + Z^{i_1}_{i_1} Z^{i_2}_{i_3} Z^{i_3}_{i_2} + Z^{i_1}_{i_2} Z^{i_2}_{i_3} Z^{i_3}_{i_1} + Z^{i_1}_{i_3} Z^{i_2}_{i_1} Z^{i_3}_{i_2}) \\ &= \frac{1}{6}(\operatorname{Tr}(Z)^3 + 3 \operatorname{Tr}(Z^2) \operatorname{Tr}(Z) + 2 \operatorname{Tr}(Z^3)).\end{aligned}$$

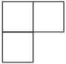
Class	σ	$\operatorname{Tr}(\sigma Z^{\otimes 3})$
1^3	$\mathbb{1}$	$\operatorname{Tr}(Z)^3$
$2\ 1$	$(1\ 2), (1\ 3), (2\ 3)$	$\operatorname{Tr}(Z) \operatorname{Tr}(Z^2)$
3	$(1\ 2\ 3), (1\ 3\ 2)$	$\operatorname{Tr}(Z^3)$

Schur polynomials 2

- IRR matrices can be easily computed by the Young-Yamanouchi symbol.

$$\Gamma_R((k,k+1))|R\rangle = \frac{1}{c_k - c_{k+1}}|R\rangle + \sqrt{1 - \frac{1}{(c_k - c_{k+1})^2}}|R_{(k,k+1)}\rangle$$

- Example: S_3



0	1	2	3	4
-1	0	1	2	
-2				

$$\chi_{(2,1)}(Z) = \frac{1}{6}\left(2\mathrm{Tr}(Z)^3 - 2\mathrm{Tr}(Z^3)\right)$$

$$\Gamma_{\begin{smallmatrix}3&2\\1&\end{smallmatrix}}(1) = \Gamma_{\begin{smallmatrix}3&1\\2&\end{smallmatrix}}(1) = 1 \qquad \Gamma_{(2,1)}(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \chi_{(2,1)}(1) = \mathrm{Tr}\left(\Gamma_{(2,1)}(1)\right) = 2$$

$$\Gamma_{\begin{smallmatrix}3&2\\1&\end{smallmatrix}}(1,2) = -\frac{1}{2}\Gamma_{\begin{smallmatrix}3&2\\1&\end{smallmatrix}} + \frac{\sqrt{3}}{2}\Gamma_{\begin{smallmatrix}3&1\\2&\end{smallmatrix}} \qquad \Gamma_{(2,1)}(1,2) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \to \chi_{(2,1)}(1,2) = \mathrm{Tr}\left(\Gamma_{(2,1)}(1,2)\right) = 0$$

$$\Gamma_{\begin{smallmatrix}3&1\\2&\end{smallmatrix}}(1,2) = \frac{1}{2}\Gamma_{\begin{smallmatrix}3&1\\2&\end{smallmatrix}} + \frac{\sqrt{3}}{2}\Gamma_{\begin{smallmatrix}3&2\\1&\end{smallmatrix}} \qquad \Gamma_{(2,1)}(2,3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to \chi_{(2,1)}(2,3) = \mathrm{Tr}\left(\Gamma_{(2,1)}(2,3)\right) = 0$$

$$\Gamma_{(2,1)}(1,2,3) = \Gamma_{(2,1)}(1,2)\Gamma_{(2,1)}(2,3) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \to \chi_{(2,1)}(1,2,3) = \mathrm{Tr}\left(\Gamma_{(2,1)}(1,2,3)\right) = -1$$

$$\chi_{(2,1)}(1,3,2) = \mathrm{Tr}\left(\Gamma_{(2,1)}(1,3,2)\right) = -1$$

Schur polynomials 3

- Why is Schur good?

  - The Schurs diagonalize the free field two-point functions.
  - Complete but not overcomplete basis for the local gauge invariant operators
  Thus, Finite N contratint(Trace relation) encoded: if Young diagram has more than N rows, it automatically becomes zero. So, counting problem is well-defined.

$$\chi_{\yng(1,1,1)}(Z) = \frac{1}{6}\left((\mathrm{Tr}\, Z)^3 - 3\,(\mathrm{Tr}\, Z)\left(\mathrm{Tr}\, Z^2\right) + 2\mathrm{Tr}\, Z^3 \right)$$

- Beyond the 1/2-BPS sector, we need to put some different types of “letters” into a “word”. For the cases, we should generalize the Schur polynomial to a multi-matrix version. It is called “restricted Schur polynomials”.

$$\chi_{R,(r,s)\alpha\beta}(Z,Y) = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma) \mathrm{Tr}(Z^{\otimes n} \otimes Y^{\otimes m})$$

$$P_{R,(r,s)} = \frac{d_r d_s}{n!m!} \sum_{\sigma_1 \in S_n} \sum_{\sigma_2 \in S_m} \chi_r(\sigma_1)\chi_s(\sigma_2)\Gamma_R(\sigma_1\sigma_2) \qquad \chi_{R,(r,s),jk}(\sigma) = \mathrm{Tr}_R(P_{R,(r,s),jk}\Gamma_R(\sigma))$$

- R is an IRR of S_{n+m} and is associated with a Young diagram of n+m-boxes. On the other hand, the restricted character is related to the restriction of S_{n+m} to S_n x S_m.

Schur polynomials 4

- Examples

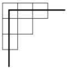
$$\chi_{\square\square;\square\otimes\square} = \mathrm{Tr}\,(Z)\mathrm{Tr}\,(X) + \mathrm{Tr}\,(ZX), \qquad \chi_{\substack{\square\\\square};\square\otimes\square} = \mathrm{Tr}\,(Z)\mathrm{Tr}\,(X) - \mathrm{Tr}\,(ZX)$$

$$\chi_{\square\square\square;\square\square\otimes\square} = \frac{1}{2}\left[\mathrm{Tr}\,(Z)^2\mathrm{Tr}\,(X) + \mathrm{Tr}\,(Z^2)\mathrm{Tr}\,(X) + 2\mathrm{Tr}\,(ZX)\mathrm{Tr}\,(Z) + 2\mathrm{Tr}\,(Z^2X)\right],$$

$$\chi_{\substack{\square\\\square\\\square};\substack{\square\\\square}\otimes\square} = \frac{1}{2}\left[\mathrm{Tr}\,(Z)^2\mathrm{Tr}\,(X) - \mathrm{Tr}\,(Z^2)\mathrm{Tr}\,(X) - 2\mathrm{Tr}\,(ZX)\mathrm{Tr}\,(Z) + 2\mathrm{Tr}\,(Z^2X)\right].$$

- Computing restricted characters is generally technical. There are two ways to compute those: strand diagrams and Casimir & projectors. But fortunately we need not to know it explicitly in our main calculations. An important fact is that the RSP basis again diagonalizes the free field two-point function.

$$\langle \chi_{R,(\vec{r},\vec{s}),\alpha_1\alpha_2}(\phi^I,\psi^J)\chi_{T,(\vec{t},\vec{u}),\beta_1\beta_2}(\phi^I,\psi^J)^\dagger\rangle = \delta_{RT}\prod_{I=1}^{b}\frac{\delta_{r_It_I}}{\mathrm{hooks}_{r_I}}\prod_{J=1}^{f}\frac{\delta_{s_Ju_J}}{\mathrm{hooks}_{s_J}} f_R \mathrm{hooks}_R$$



$$\text{hook lengths} = \begin{array}{|c|c|c|}\hline 5 & 3 & 1 \\ \hline 3 & 1 & \\ \cline{1-2} 1 & & \\ \cline{1-1}\end{array}\,. \qquad d_R = \frac{n!}{\prod_{x\in R}\mathrm{hook}(x)} \equiv \frac{n!}{\mathrm{hooks}_R}$$

Dual gauge operators of black holes 1

- With the restricted Schur polynomials, one can make a heavy operator with O(N^2) and introduce all super YM fields into the operators.

- What additional conditions should be applied to claim that it is a black hole?

- In the case of a 1/16-BPS black hole, we can think about it by considering the field contents that make up the sector, but what about a general non-BPS black hole?

- Perhaps one of the most important properties of black holes that can be realized by dual operators is state counting.

- To explain the enormous entropy of a black hole, we need some kind of ensemble, but what could it be?

- One of the natural idea is to count number of all Young diagrams with size n.

Dual gauge operators of black holes 2

- If the counting is correct, there should be order e^{N^2} distinct operators with the Young diagrams of $O(N^2)$ boxes.

- However, counting Young diagrams is equivalent to the number of partitions. According to Hardy/Ramanujan formula, it grows like $e^{\sqrt{n}}$ in large n. Thus, by varying the Young diagram labels, we get $\sim e^{N}$. This is too few to have N^2 entropy.

- What other possibilities could give correct black hole microstate counting?

- In the restricted Schur polynomials, there exists a notion of multiplicies called Littlewood-Richardson(LR) coefficients.

- If we want to do state counting with such multiplicity labels, we have to change the question: For what kind of Young's diagram does the condition allow the maximum LR coefficients?

Entropy of BHs (1)

- To figure the LR coefficients out, it is enough to consider two-matrix model sector. Also, this is reasonable because the sector seems to have some BH like properties from the matrix model.

- When we consider merging among several Young diagrams, there exists the multiplicity labels.

$$\text{Consider } R = \yng(3,2,1) \qquad (\yng(2,1), \yng(1)\,\yng(1)\,\yng(1)), (\yng(1,1,1), \yng(2,1)), (\yng(3), \yng(2,1))$$

Thus, upon restricting to $S_3 \times S_3 \subset S_6$ we find that R returns the following irreps

$$(\yng(2,1), \yng(3)) \oplus (\yng(2,1), \yng(2,1)) \oplus (\yng(2,1), \yng(2,1)) \oplus (\yng(2,1), \yng(1,1,1)) \oplus (\yng(1,1,1), \yng(2,1)) \oplus (\yng(3), \yng(2,1)).$$

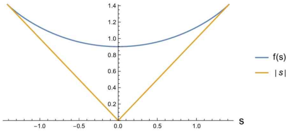
Note that there are two copies of $(\yng(2,1), \yng(2,1))$. The Littlewood-Richardson number counts this multiplicity. In our case $g_{\yng(2,1)\,\yng(2,1)\,\yng(3,2,1)} = 2$. The Littlewood-Richardson number g_{rst} is an integer.

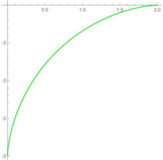
- If we want to count the Littlewood-Richardson coefficients, this means we are fixing the Young diagrams (the brane configuration) and reproduce the required growth to obtain the number of black hole states completely from the multiplicity labels (open string configurations). This is similar to the Strongminger-Vafa counting.

Entropy of BHs (2)

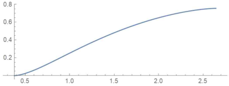
- Back to the question, what is the LR coefficient in large n? This is a mathematical question. Is there a limit shape which maximizes LR coefficient? ['77 Logan, Shepp] ['77 Vershik, Kerov]

- Vershik, Kerov, Logan and Shepp showed that it is given as a limit curve called VKLS shape. It is almost triangular. Precisely, if R, r and s are all VKLS types, we have the maximal LR coefficients.





- This is valid for n < N. On the other hand, with finite N constraint, the curve should be truncated. This changes the behavior of entropy.



Entropy of BHs (3)

- In each case, what are the maximal LR coefficients in large n?
- ['18 Igor, Panova, Yeliussizov]

Littlewood-Richardson(LR) coefficient $c(n, m_1)$

- VKLS $\quad c(n, m_1) = 2^{\frac{n}{2}}$

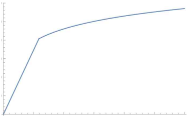
$$S = \log c^2(n, m_1) \simeq N^2 \log 2$$

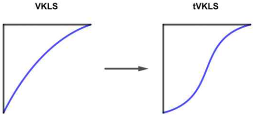
- Truncated VKLS

$$C_N(n) = (n+1)^{\frac{1}{2}N^2}$$

$$S = \log C_N^2(n) = N^2 \log E$$

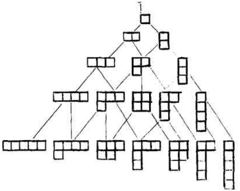
- Entropy(small to large BHs?)





Entropy of BHs (4)

- Furthermore, at large N, we expect that BHs are typical states.
- At small number of N, gravitons and BHs are of the same order. However, at large N, we definitely believe that most of states must be BHs.
- Besides the maximal multiplicity labels, the VKLS shape also defines such a typicality. Namely, the VKLS corresponds to the singular distribution.
- For a very large number of boxes in the Young diagram so that asymptotic partition theory applies, if we randomly pick a Young diagram from the space of all allowed diagrams, we will always pick the VKLS shaped one.



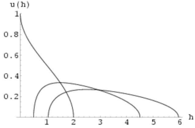
Relation to work by Dutta and Gopakumar 1

- Partition function on S3 x S1 and character expansion

$$Z(\beta) = \int [dU] \exp\left[\sum_{n=1}^{\infty} \frac{a_n(T)}{n} \mathrm{Tr}(U^n)\mathrm{Tr}(U^{\dagger n})\right] \qquad a_n(T) = z_{\mathrm{B}}(x^n) + (-1)^{n+1} z_{\mathrm{F}}(x^n)$$

$$= \sum_{\vec{k}} \frac{\prod_j a_j^{k_j}}{z_{\vec{k}}} \sum_R \left[\chi_R(C(\vec{k}))\right]^2$$

$$Z(\beta) = \sum_{k=0}^{\infty} \sum_R \frac{1}{k!} \left[d_R(S_k)\right]^2 a_1^k \qquad h_i = n_i + N - i$$



- Hook length formula and Saddles

$$n(x) = \frac{n_i}{N}, \qquad h(x) = \frac{h_i}{N}, \qquad x = \frac{i}{N}$$

$$Z = \int [dh(x)] e^{-N^2 S_{\mathrm{eff}}}$$

$$h(x) = n(x) + 1 - x$$

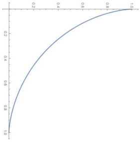
$$S = \int_0^1 dx \int_0^1 dy \ln |h(x) - h(y)| - 2\int_0^1 dx h(x) \ln h(x) + k' \ln(a_1 k') + k' + 1$$

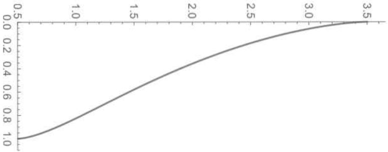
$$u(h) = -\frac{\partial x(h)}{\partial h}$$

$$k' \equiv \int_0^1 dx h(x) - \frac{1}{2} \qquad u(h) = \frac{1}{\pi} \arccos\left[\frac{h-1}{2\xi} + \frac{\left(\xi - \frac{1}{2}\right)^2}{2\xi h}\right] \qquad \tilde{u}(h) = \frac{1}{\pi} \arccos\left[\frac{h-1}{2\xi}\right]$$

Relation to work by Dutta and Gopakumar 2

- What do the saddle solutions mean in Young diagram?
- For each saddles, one can replot them as Young diagrams (x(h), n(h)).
- We found they correspond to VKLS and tVKLS.





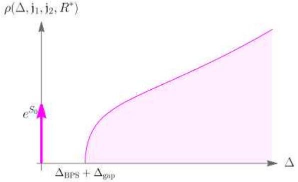
- Furthermore, we showed the entropy in high temperatute limit is given as

$$N^2 \log E$$

- This is a nice check for our conjecture.

Statistical data of BPS and near BPS black holes

- ['22 Boruch, Heydeman, Iliesiu, Turiaci]



- The gap is related to the fact that the Wilsonian low energy effective action should have a Schwarzian mode.

- The gap was computed from the gravitational path integral and has the size of 1/N^2 ~ 1/S.

- Can we show this gap from N=4 SYM?

Dilatation Operator 1

- The one-loop dilatation operator was computed as ['04 Beisert]

$$H = \sum_{A,B,C,D} \mathcal{C}^{AB}_{CD} (-)^{AB} (-)^{AD} \mathrm{Tr} \left[W_C, \check{W}^A \right\} \left[W_D, \check{W}^B \right\}$$

$$H \mid W_1 \times W_2 \rangle = (\text{number}) \ \times \mid W_3 \times W_4 \rangle$$

$$c_{n,n12,n21} = (-1)^{1+n12n21} \frac{\Gamma\left(\frac{n12+n21}{2}\right) \Gamma\left(1 + \frac{1}{2}(n - n12 - n21)\right)}{\Gamma\left(1 + \frac{1}{2}n\right)}$$

- We need to understand the action of dilatation operator on the restricted Schur polynomials. For simplicity, let's consider the SU(2) sector.

$$D \ = \ -\frac{g_{YM}^2}{16\pi^2} \mathrm{Tr} \left[Z, Y \right] \left[\check{Z}, \check{Y} \right]$$

- In the planar limit, it generates hopping of magnon in the spin-chain. On the other hand, in our problem, it remove a box from Young diagram and attach it.

Dilatation operator 2

- The matrix element was calculated before. ['10-'12 dMK et al.]
- Normalized restricted Schur polynomial

$$\chi_{R,(r,s)jk}(Z,Y) = \sqrt{\frac{f_R \,\mathrm{hooks}_R}{\mathrm{hooks}_r \,\mathrm{hooks}_s}} O_{R,(r,s)jk}(Z,Y)$$

- Action of D

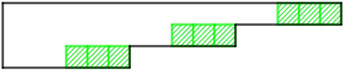
$$DO_{R,(r,s)jk}(Z,Y) = \sum_{T,(t,u)lq} N_{R,(r,s)jk;T,(t,u)lq} O_{T,(t,u)lq}(Z,Y) \qquad c_{RR'} = N - i + j$$

$$\begin{aligned} N_{R,(r,s)jk;T,(t,u)lq} \quad &= -g_{YM}^2 \sum_{R'} \frac{c_{RR'} d_T n m}{d_{R'} d_t d_u (n+m)} \sqrt{\frac{f_T \,\mathrm{hooks}_T \,\mathrm{hooks}_r \,\mathrm{hooks}_s}{f_R \,\mathrm{hooks}_R \,\mathrm{hooks}_t \,\mathrm{hooks}_u}} \times \\ &\times \mathrm{Tr}\Big(\Big[\Gamma_R((1,m+1)), P_{R\to(r,s)jk} \Big] I_{R'T'} \Big[\Gamma_T((1,m+1)), P_{T\to(t,u)ql} \Big] I_{T'R'} \Big) \end{aligned}$$

- I_{R'T'} is an intertwiner which maps from R' to T' and we used the projectors which are related to the restriction of the symmetric group. To get this formally, we need to do rather long and complicated calculations.

Diagonalization

- Let us assume distant corners approximation. This does not apply when dealing with black holes, but they are good for understanding more simplified situations.



- It simplifies the diagonalization. The eigenfunction is called Gauss graph operator which are combination of restricted Schur polynomials.
- Here, there are larger numbers of Z's than those of Y's. So, this is a perturbation of 1/2-BPS operator which is a system of giant graviton branes with each row being a single brane. On the other hand, Y's are interpreted as open strings that dress the giant graviton branes.
- Since the worldvolume of each brane is compact, the total number of strings ending on the brane must be equal to the that starting from the brane.

Gauss Graph Operators and Eigenvalues

- Eigenvalue equation

$$DO_{R,r_1,\sigma}(Z,Y) = -\frac{g_{YM}^2}{16\pi^2}\sum_{i<j=1}^{N} n_{ij}\Delta_{ij}O_{R,r_1,\sigma}(Z,Y)$$

- Eigenvalues

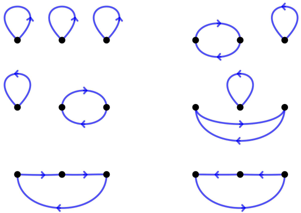
$$\Delta_{ij} = \Delta_{ij}^{+} + \Delta_{ij}^{0} + \Delta_{ij}^{-}$$

$$\Delta_{ij}^{0}O_{R,(r_1,r_2)\alpha_1\alpha_2} = -(2N + R_i + R_j)O_{R,(r_1,r_2)\alpha_1\alpha_2}$$

$$\Delta_{ij}^{+}O_{R,(r_1,r_2)\alpha_1\alpha_2} = \sqrt{(N+R_i)(N+R_j)}\,O_{R_{ij}^{+},(r_{1,ij}^{+},r_2)\alpha_1\alpha_2}$$

$$\Delta_{ij}^{-}O_{R,(r_1,r_2)\alpha_1\alpha_2} = \sqrt{(N+R_i)(N+R_j)}\,O_{R_{ij}^{-},(r_{1,ij}^{-},r_2)\alpha_1\alpha_2}$$

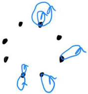
- Example



BPS States and Near-BPS States

- BPS operators

$$DO_{BPS} = 0$$



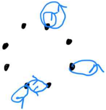
- This implies n_{ij}=0 . So, all of the edges in the Gauss graph begin and return to the same node.

- The state with the smallest anomalous dimension has one pair of edges draped between a pair of nodes. Our goal is to compute the energy of this state.

- Let's consider a graph with edges draped between nodes i and i+1, and all others begin and terminate at the same node. Then, we have

$$D\Psi(Z,Y) = E_{\text{gap}}\Psi(Z,Y)$$

$$DO_{R,r_1,\sigma}(Z,Y) = -\frac{g_{YM}^2}{8\pi^2}\Delta_{i\,i+1}O_{R,r_1,\sigma}(Z,Y)$$



- The eigenfunction is a linear combination of Gauss graph operators.

Energy Gap 1

- By considering N >> a and a continuum limit through

$$x = \frac{a}{K}$$

- We get

$$E_{\rm gap} f(x) = \frac{g_{YM}^2}{8\pi^2}\Bigg(-\frac{N+\bar{R}}{K^2}\left(\frac{d^2 f}{dx^2} + O(K^{-2})\right) + \frac{K^2 x^2}{(N+\bar{R})}(f(x) + O(K^{-2}))\Bigg)$$

$$\begin{aligned} E_{\rm gap} f(x) &= \frac{g_{YM}^2}{8\pi^2}\bigg(-\frac{N+\bar{R}}{K^2}\frac{d^2 f}{dx^2} + \frac{K^2 x^2}{(N+\bar{R})} f(x)\bigg) \\ &= \frac{g_{YM}^2}{8\pi^2}\bigg(-\frac{1}{2m}\frac{d^2 f}{dx^2} + \frac{k x^2}{2} f(x)\bigg) \end{aligned}$$

- Thus, we obtain

$$\omega^2 = \frac{k}{m} = \Big(\frac{2N+2\bar{R}}{K^2}\Big)\Big(\frac{2K^2}{(N+\bar{R})}\Big) = 4 \qquad E_{\rm gap} \sim \frac{\lambda}{N}$$

- This is larger than what we wanted to obtain ~ 1/N^2.

Energy Gap 2

- However, this is not strange. We argued that Young diagrams about BHs should be VKLS shape. In the VKLS, each corners are surely of order 1.
- Thus, our distant corners approximation cannot capture the correct property.
- In string side, the distant corners approximation means a kind of Coulomb branch. Here, each brane is distant from each other. We need to count YD labels which are matched with 1/S gap.
- Because each row in Young diagram is dual giant gravitons, the total representation is a boundstate of giant gravitons.
- BHs are not just that. We need more delicate interactions between branes. Thus, we have to diagonalize the problem which has the order 1 corners.
- This is very recently solved in mathematics community['22 Finkelberg, Postnikov, Schechtman] and we are still in progress.

Discussion

- There are some interesting angles and some of them are also in progress.
- One of the most important directions is to construct 1/16 BPS BH operators from restriced Schur basis.
- There are technically difficult parts in finding black holes through Q-cohomology problem, which are in the trace relation and Q-exactness check. Furthermore, there must be an ensemble existence to explain the entropy of black holes.
- Existing approaches seem to lack both of these. The process of obtaining dual operators is not systematic, and it is difficult to ask questions such as the entropy of a black hole from the operators obtained in this way.
- We expect the restricted Schur techniques would provide a systematic approach. Studying Q-cohomology problem by using restricted Schurs is in progress.