New Integrable RG Flows with Parafermions

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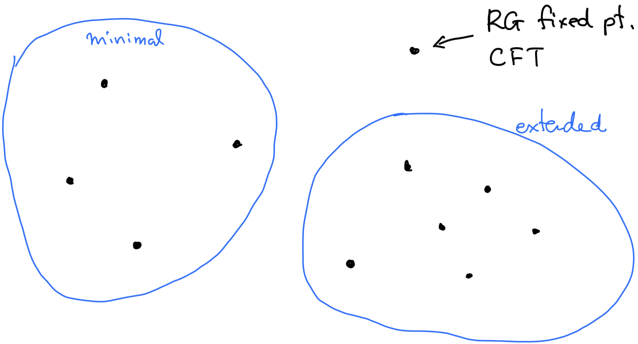
Based on

Z. Bajnok & CA [arXiv:2407.06582]

Integrability, Duality and Related topics 2024, APCTP

Introduction

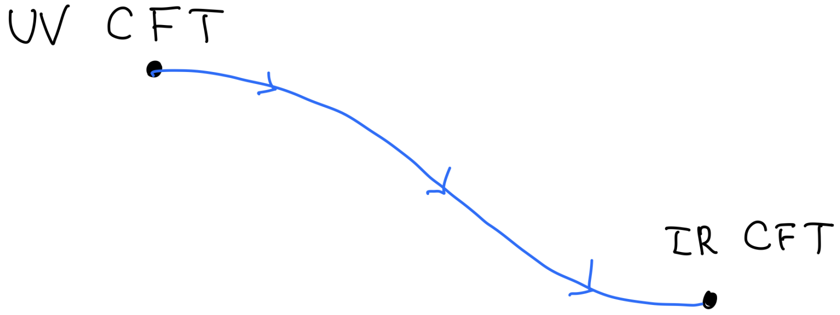
2D CFTs



CFTs are special

- Correlation functions exactly calculable in terms of CFT data (c, Δ_i)
- Pretty much classified (Minimal, WZW, super, W-algebra, ...)
- (Ex) Minimal (unitary) CFTs: $\mathcal{M}_p,\ c_p = 1 - \frac{6}{p(p+1)}$

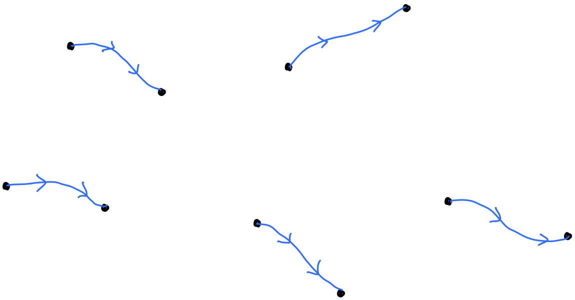
Special 2D QFTs



QFTs which are both UV and IR complete

- Exact RG flows: UV CFT+relevant fields $\rightarrow$ IR CFT + irrelevant fields
- Need non-perturbative methods, e.g. integrability
- Exactly solvable in all scale

Can we classify these UV-IR complete QFTs?

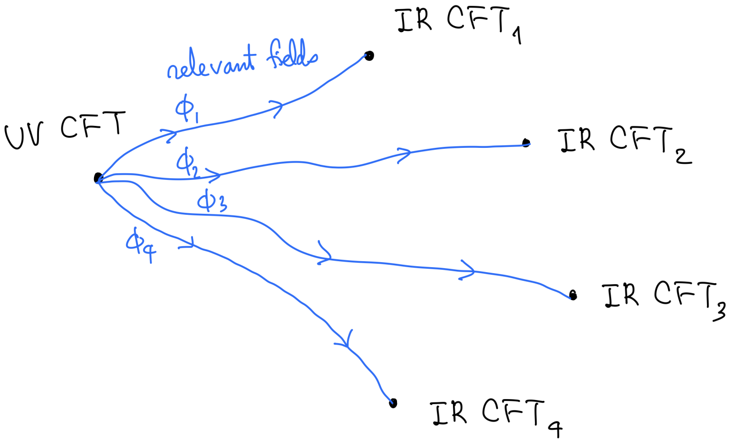


No systematic approach: many such pairs are known mainly by conjectures

- Non-invertible symmetries may help [Nakayama's talk]
- But mostly effective QFTs and focus on both CFTs only
- Identifications by exact S-matrices or explicit Lagrangians are needed

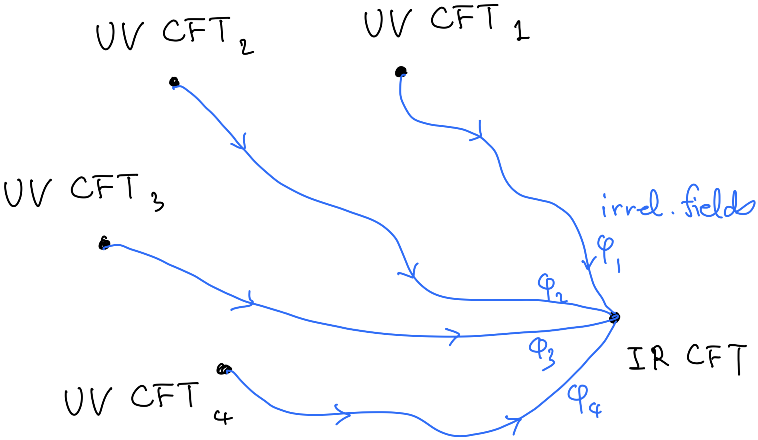
Different RG flows from a UV CFT

Different relevant fields may lead to different IR CFTs (only finite cases)

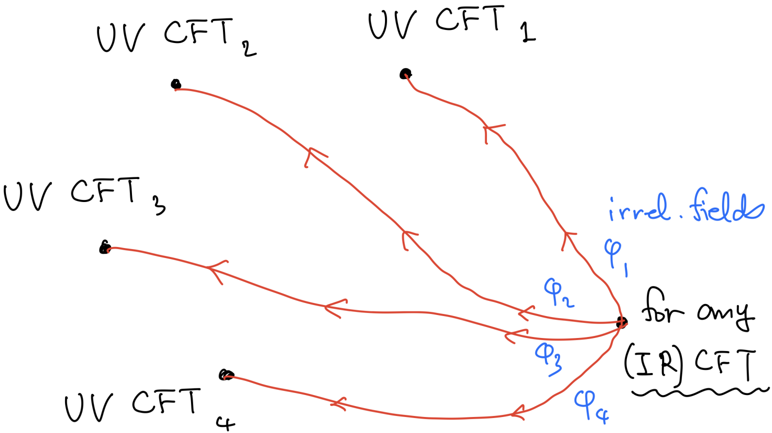


Different RG flows into a IR CFT

RG flows into a IR CFT in different irrelevant fields directions (infinite possibilities)



New approach: Inverse RG flows



- Integrability approach: RG flows from IR to UV
- Complete classification if all UV CFTs are found for each (IR) CFT
- Challenge: How to identify UV CFTs and relevant fields?

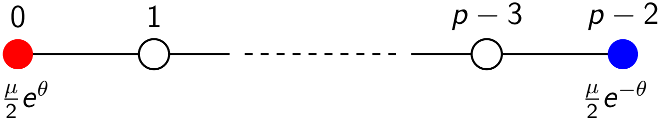
Inverse RG flows

Conventional Approach (top down)

- UV CFT+a relevant field $\longmapsto$ IR CFT+ irrelevant fields
  - Only a special relevant field maintains the integrability
  - Common in CMP since physics at low T is important to find non-trivial (Wilson-Fisher) fixed points

- (Ex) Zamolodchikov Flows originally based on conjectured TBA

$$\mathcal{M}_{p+1} + \lambda\Phi_{1,3} \to \mathcal{M}_p + \lambda'\Phi_{3,1}, \quad p = 3, 4, \cdots$$



- Many more RG flows have been conjectured
  - have been guessed based on conjectured TBA or NLIE
  - Lagrangians / exact S-matrices are missing

New approach (bottom up) [CA, A. LeClair (2022)]

- IR CFT+ irrelevant fields $\longmapsto$ UV CFT+a relevant field
- Natural since S-matrices are defined in the IR (infinite volume)
- Common in HEP where UV complete theory is being searched (GUT, SUSY, Superstring, ...)
- Use $T\overline{T}$ as a ladder

$T\overline{T}$ deformations

- Very active developments [Tateo,Zamolodchikov,...]
- Energy-momentum tensor T_2
- All higher conserved charges $= \{ \; [T\overline{T}]_s \; \}$

$$[T\overline{T}]_s = T_{s+1}\overline{T}_{s+1} - \Theta_{s-1}\overline{\Theta}_{s-1}, \quad \partial_{\bar{z}} T_{s+1} = \partial_z \Theta_{s-1}$$

- Preserve integrability
- Exact results possible for even non-integrable theories

Space of 2D IQFTs [Smirnov-Zamolodchikov (2017)]

- Expands the integrable space in infinite dimensions
- If the mother theory is integrable, new integrable QFTs

$$\text{New IQFTs} = \text{an IQFT} + \sum_{s=1}^{\infty} \alpha_s [T\overline{T}]_s$$

- Exact S-matrices are given by additional CDD factors

$$\mathrm{S}(\theta) = \prod_{s=1}^{\infty} e^{2i\alpha_s M^{2s} \sinh(s\theta)} \cdot \mathrm{S}_0(\theta)$$

Swampland (Hagedorn singularity) to cross for UV

- Burgers Equation ($s = 1$) $[CFT + \alpha_1 [T\overline{T}]_1]$

$$\partial_\alpha E + E \partial_R E = 0 \quad \rightarrow \quad c_{\text{eff}}(R) = \frac{2c_0}{1 + \sqrt{1 + \frac{2\pi\alpha_1 c_0}{3R^2}}}$$

  with square root singularity at $R_c = \sqrt{\frac{2\pi|\alpha_1|c_0}{3}}$

- Singularity occurs also for each $[T\overline{T}]_s$ with $s \geq 1$
- We show that the singularities can be avoided if we fine-tune all α_s

S-matrices of CFTs from a massless limit

- Consider "massive" integrable deformation of a CFT e.g.

$$[G]_k \equiv \frac{G_1 \otimes G_k}{G_{k+1}} + \lambda\, \Phi_{\text{least rel}}$$

- Integrable with exact S-matrix S_0 [Bernard, LeClair, CA (1990)]
- Take $\lambda \to 0^-$ or $M \to 0$ limit
  - rescale the rapidity

$$M \to 0 \ \&\ \theta = \pm\Lambda + \hat{\theta} \quad \text{with finite} \quad Me^{\Lambda} = \mu$$

  - R and L particles appear depending on $\pm$

$$(R) : E = P = \frac{\mu}{2} e^{\hat{\theta}}, \qquad (L) : E = -P = \frac{\mu}{2} e^{-\hat{\theta}}$$

  - S-matrices between RR, LL are the same $\mathsf{S}^{RR}(\hat{\theta}) = \mathsf{S}^{LL}(\hat{\theta}) = \mathsf{S}_0(\hat{\theta})$
  - S-matrices between RL, LR are trivial since $\theta_{12} \to \pm\infty$

S-matrices for deformed CFTs by $T\overline{T}$'s

- CDD factors become trivial in RR, LL sectors

$$\prod_{s=1}^{\infty} e^{2i\alpha_s M^{2s}\sinh(s\theta)} \to 1 \quad \text{as} \quad M \to 0$$

- CDD factors become non-trivial in RL, LR sectors

$$\prod_{s=1}^{\infty} e^{2i\alpha_s M^{2s}\sinh(s\theta)} \to \prod_{s=1}^{\infty} e^{\pm i\alpha_s \mu^{2s} e^{\pm s\hat{\theta}}} \equiv \mathsf{S}_{\mathsf{CDD}}(\hat{\theta})$$

- Summary

$$\boxed{\mathsf{S}^{RR}(\theta) = \mathsf{S}^{LL}(\theta) = \mathsf{S}_0,\ \mathsf{S}^{RL}(\theta) = \mathsf{S}^{LR}(-\theta) = \mathsf{S}_{\mathsf{CDD}}(\theta)}$$

How to restrict S_{CDD}?

- Find all possible S_{CDD} for a given S_0 which satisfy
- UV completeness
  - No Hagedorn singularities
  - $c_{\rm eff}$ in the UV limit is **finite, rational**
- Crossing-Unitarity

$$S_{CDD}(\theta)S_{CDD}(\theta + i\pi) = 1$$

from which S_{CDD} are arbitrary products of only two factors

$$S^{(1)}_{CDD} = -\tanh\left(\frac{\theta-\beta}{2} - \frac{i\pi}{4}\right), \quad S^{(2)}_{CDD} = \frac{\sinh\theta - i\sin\beta}{\sinh\theta + i\sin\beta}$$

- All α_s are fixed exactly in terms of β's
- We can work with UV limit of TBA, namely, "plateaux equations"

Focus on "diagonal" S-matrices theories

- S-matrices for $[G]_1$ have been known [Zamolodchikov,...]
- A CFT can have different S-matrices depending on the choice of the integrable relevant field perturbations
  (ex) Ising CFT by $G = su(2), E_8$
- Plateaux equations

$$x_a^R = \prod_{b=1}^{r}(1 + x_b^R)^{k_{ab}} \prod_{b=1}^{r}(1 + x_b^L)^{\tilde{k}_{ab}}$$

  where exponents are integrals of logarithmic derivatives of S-matrices S_0^{ab} and $\mathrm{S}_{\mathrm{CDD}}^{ab}$ ($\tilde{k}_{ab} = \frac{1}{2}, 1$ or sum of them)
- c_{eff} in terms of some dilog identities

Results: UV CFTs which can flow into Ising CFT $\mathcal{M}_3$

- From the Plateaux equations, we find only

$$c_{\text{eff}} = \frac{7}{10},\ \frac{21}{22},\ \frac{3}{2},\ \frac{15}{2},\ \frac{31}{2}$$

  - "$\frac{7}{10}$" is the Zamolodchikov flow $\mathcal{M}_4 \to \mathcal{M}_3$ ($[su(2)]_2 \to [su(2)]_1$)
  - "$\frac{21}{22}$" is the coset flow $[E_8]_2 \to [E_8]_1$
  - "$\frac{3}{2}$" is the massless SShG model which flows from super-Liouville theory (later) [Kim, Rim, Zamolodchikov,CA (2002)]
  - New: "$\frac{31}{2}$" is supposed to be a flow from E_8 WZW with level 2
  - New: "$\frac{15}{2}$" (?)
- We analyzed other group $G = su(3), su(4)$ etc to classify all possible UV CFTs based on central charges
- But central charges are **not enough** to identify the QFTs

Identifying UV complete QFTs

RSOS (non-diagonal) scattering theory

- Consider $k = p - 2$ with $G = su(2)$

$$\mathcal{M}_p + \lambda\Phi_{1,3}, \qquad \lambda < 0$$

- Particle spectrum: massive kinks

$$a \Bigg\uparrow b \;=\; K_{ab}(\theta), \qquad a, b = 0, \frac{1}{2} \cdots, \frac{p}{2} - 1, \quad \text{with} \quad |a - b| = \frac{1}{2}$$

- S-matrix of kinks $S^{[\mathrm{p}]}_{\mathrm{RSOS}}(\theta)$: non-diagonal [Bernard, LeClair (1990)]

$$K_{ab}(\theta_1)K_{bc}(\theta_2) \quad \rightarrow \quad K_{ad}(\theta_2)K_{dc}(\theta_1)$$

$$\mathsf{S}_p(\theta)^{ab}_{dc} = U(\theta)\;(X^{ac}_{db})^{\frac{i\theta}{2\pi}}\left[(X^{ac}_{db})^{\frac{1}{2}}\sinh\left(\frac{\theta}{p}\right)\,\delta_{db} + \sinh\left(\frac{i\pi - \theta}{p}\right)\,\delta_{ac}\right]$$

Massless S-matrices [Fendley, Saleur, Zamolodchikov ('93)]

- Massless limit $\lambda \to 0^-$

$$\mathsf{S}_p^{RR}(\theta) = \mathsf{S}_p^{LL}(\theta) = \mathsf{S}_p(\theta)$$

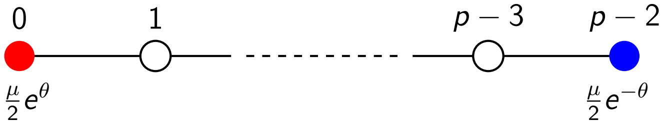
- YBE and Crossing-Unitarity relations determine

$$\mathsf{S}_p^{RL}(\theta) = \mathsf{S}_p^{LR}(-\theta) \propto \mathsf{S}_p\left(\theta - \frac{ip\pi}{2}\right)$$

  for the S-matrices of the IR theory $\mathcal{M}_p + \lambda' \Phi_{3,1}$

- Derived the conjectured TBA (partially) shown before

$$\epsilon_a(\theta) = \frac{\mu R}{2}(\delta_{a0} e^{\theta} + \delta_{a,p-2} e^{-\theta}) - \sum_{b=0}^{p-2} \mathbb{I}_{ab} \varphi \star \log[1 + e^{-\epsilon_b}](\theta)$$



TBA for $[T\overline{T}]_s$ deformed minimal CFT $\mathcal{M}_p$

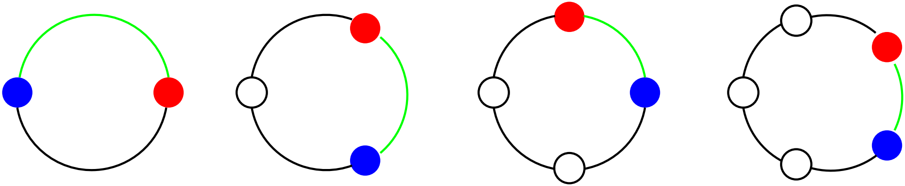
- $[T\overline{T}]_s$ deformations

$$\mathcal{M}_p + \lambda' \, \Phi_{3,1} + \sum_{s \geq 1} \alpha_s \, [T\overline{T}]_s$$

- Introduce CDD factors to RL, LR sectors

$$\tilde{\mathsf{S}}_p^{RL}(\theta) = \mathsf{S}_{\mathrm{CDD}} \cdot \mathsf{S}_p^{RL}(\theta)$$

- Being diagonal, $\mathsf{S}_{\mathrm{CDD}}$ introduces additional kernels between R & L



UV complete theories

- Plateaux equations

$$
\begin{aligned}
x_n &= (1+x_{n-1})^{1/2}(1+x_{n+1})^{1/2}, \quad n=1,\cdots,p-3, \\
x_0 &= (1+x_1)^{1/2}(1+x_{p-2})^{\tilde{k}}, \quad x_{p-2}=(1+x_{p-3})^{1/2}(1+x_0)^{\tilde{k}}
\end{aligned}
$$

- UV complete only with $\tilde{k}=\frac{1}{2}$ which means

$$
S^{(1)}_{\mathrm{CDD}} = -\tanh\left(\frac{\theta-\beta}{2}-\frac{i\pi}{4}\right) \quad \rightarrow \quad \varphi_{RL}=\frac{1}{\cosh(\theta-\beta)}
$$

- which gives UV central charge

$$
c_{\mathrm{eff}} = 3\,\frac{p-1}{p+1}
$$

For $p = 3$, massless Super sinh-Gordon model

- sinh-Gordon model with $N = 1$ super [Kim, Rim, Zamolodchikov, CA (2002)]

$$\mathcal{L} = \mathrm{Kin.} - \frac{i}{2}\psi\overline{\psi}W''(\phi) + 2\pi[W'(\phi)]^2, \quad W(\phi) = -\mu \sinh(b\phi)$$

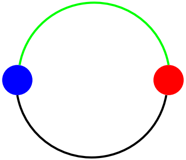
- Supersymmetry is spontaneous broken

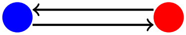
$$V(\phi) \propto [W'(\phi)]^2 = \cosh^2(b\phi) > 0$$

- massless Goldstino fermion is only spectrum in the IR limit (Ising model)
- S-matrices are simple: $S^{RR} = S^{LL} = -1, S^{RL} = \mathrm{S}_{\mathrm{CDD}}^{(2)}$
- TBA is given by

$$\begin{aligned}
\epsilon^R(\theta) &= \frac{\mu R}{2}e^{\theta} - \varphi^{RL} \star \log[1 + e^{-\epsilon^L}], \\
\varphi^{RL}(\theta) &= \frac{1}{\cosh(\theta - ia)} + \frac{1}{\cosh(\theta + ia)}, \quad a = \frac{1 - b^2}{1 + b^2}
\end{aligned}$$

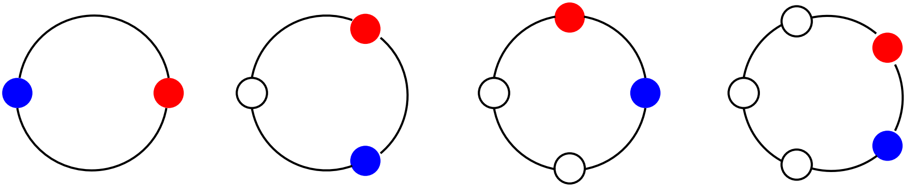
Two theories match when $\beta = 0$ and $a = 0$





- Ising model $\mathcal{M}_{p=3}$ deformed by $[T\overline{T}]_s$ with $(\beta = 0)$
  $=$ SshG model at self-dual coupling

- Conjecture: $\mathcal{M}_{p\geq 3}$ deformed by $[T\overline{T}]_s$ with $(\beta = 0)$
  $= \mathbb{Z}_{p-1}$ Parafermionic shG model at self-dual coupling



Parafermionic shG model

- PshG model

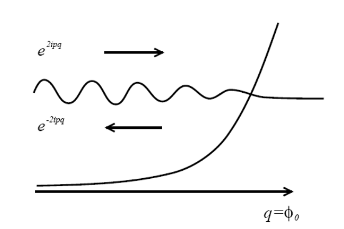
$$\mathcal{L}_{\mathrm{PShG}} = \mathcal{L}_{\mathrm{PF}} + \frac{1}{4\pi}(\partial_\mu\phi)^2 - \kappa\left(\psi_1\overline{\psi}_1 e^{2b\phi} + \eta\,\psi_1^\dagger\overline{\psi}_1^\dagger e^{-2b\phi}\right) + \cdots$$

- ψ_1 is a $\mathbb{Z}_k$ PF with $\Delta = 1 - \frac{1}{k}$ $(k \equiv p - 1)$
- Fractional SUSY
  - $\eta = 0$ gives Parafermionic LFT [Baseilhac, Fateev (1998)]
  - $\eta = 1$ massive phase
  - $\eta = -1$ massless phase where FSUSY is spontaneously broken
- These types of QFTs compute c_{eff} from momentum quantization using Reflection amplitudes

Reflection amplitudes of LFTs

- Primary fields $e^{2\alpha\phi}$ and $e^{2(Q-\alpha)\phi}$ identified upto constant

$$e^{2(\frac{Q}{2}+ip)\phi} = \mathcal{R}(p)e^{2(\frac{Q}{2}-ip)\phi}, \qquad \alpha = \frac{Q}{2} + ip,\ Q = b + \frac{1}{kb}$$

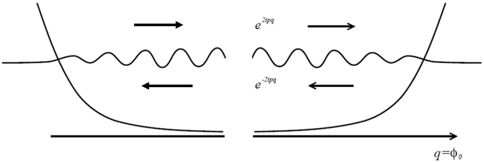


- This amplitude has been computed in the PLFT [Baseilhac, Fateev (1998)]

Momentum quantization condition

- the perturbation introduces another wall

$$\delta^{(k)}(p) = \pi + 4Qp \ln \frac{R}{2\pi} = \sum_{n=\mathrm{odd}} \delta_n p^n, \quad \mathcal{R}(p) = e^{i\delta^{(k)}(p)}$$



- Scaling function can be obtained from

$$\begin{aligned} c_{\mathrm{eff}} &= \frac{3k}{k+2} - 24p^2 + \mathcal{O}(R) \\ &= \frac{3k}{k+2} - \frac{3\pi^2}{2Q^2 \ln^2 x} - \frac{3\pi^2\delta_1}{4Q^3 \ln^3 x} - \frac{9\pi^2\delta_1^2}{32Q^4 \ln^4 x} - \frac{3(2\pi^2\delta_1^3 + \pi^4\delta_3)}{64Q^5 \ln^5 x} + \ldots \quad (x = \tfrac{R}{2\pi}) \end{aligned}$$

Reflection amplitude vs. massless TBA

- Need to solve numerically with high precision

$$c_{\text{eff}}(r) = \frac{3k}{k+2} + \sum_{n=2} \frac{c_n(k)}{(\log r)^n} + O(r)$$

- Two match very well at self-dual coupling $b = \frac{1}{\sqrt{k}}$

	c_2	c_3	c_4	c_5
$k=2$	7.402199	42.7620	185.218	714.563
	7.402203	42.7628	185.282	717.247
$k=3$	11.10332	77.8573	409.598	1924.84
	11.10330	77.8543	409.425	1919.36
$k=4$	14.8045	119.428	722.797	3898.75
	14.8044	119.4197	722.475	3892.55

Concluding Remarks

- We have found exact massless S-matrices from which we have derived TBA for new RG flows
- We have found that there is only one new RG flow to $\mathcal{M}_p$ for $p > 3$ if the CFT is in RSOS (can not exclude existence of other basis)
- We have found explicit Lagrangians (the PF sinh-Gordon models) which show new RG flows
- Can we generalize this approach to other irrelevant deformations than $T\overline{T}$'s?
- Can we find exact S-matrices behind conjectured UV-IR complete theories or claimed by non-invertible symmetries?

Thank you for attention!