S-matrix approach to AdS/CFT

Elementary level Introduction

Changrim Ahn

Ewha Womans University

Very diverse audience

Physics

- Astrophysics (**astro-ph** new, recent, find)
  includes: Astrophysics of Galaxies; Cosmology and Nongalactic Astrophysics; Earth and Planetary Astrophysics; High Energy Astrophysical Phenomena; Instrumentation and Methods for Astrophysics; Solar and Stellar Astrophysics
- General Relativity and Quantum Cosmology (**gr-qc** new, recent, find)
- High Energy Physics - Experiment (**hep-ex** new, recent, find)
- High Energy Physics - Lattice (**hep-lat** new, recent, find)
- High Energy Physics - Phenomenology (**hep-ph** new, recent, find)
- High Energy Physics - Theory (**hep-th** new, recent, find)
- Mathematical Physics (**math-ph** new, recent, find)
- Nonlinear Sciences (**nlin** new, recent, find)
  includes: Adaptation and Self-Organizing Systems; Cellular Automata and Lattice Gases; Chaotic Dynamics; Exactly Solvable and Integrable Systems; Pattern Formation and Solitons

- Numerical
- Physical

- Abstract
- Unphysical
- Rigorous

Goal

Can we find physical quantities from coupling constant?

$$F(g)$$

- not so successful except perturbations

(Ex) magnetic moment of electron

- Anomalous moment:
  - Exp.: $a_e \equiv \frac{g-2}{2} = 0.001\ 159\ 652\ 180\ 85(76)$
  - Should be determined by coupling constant

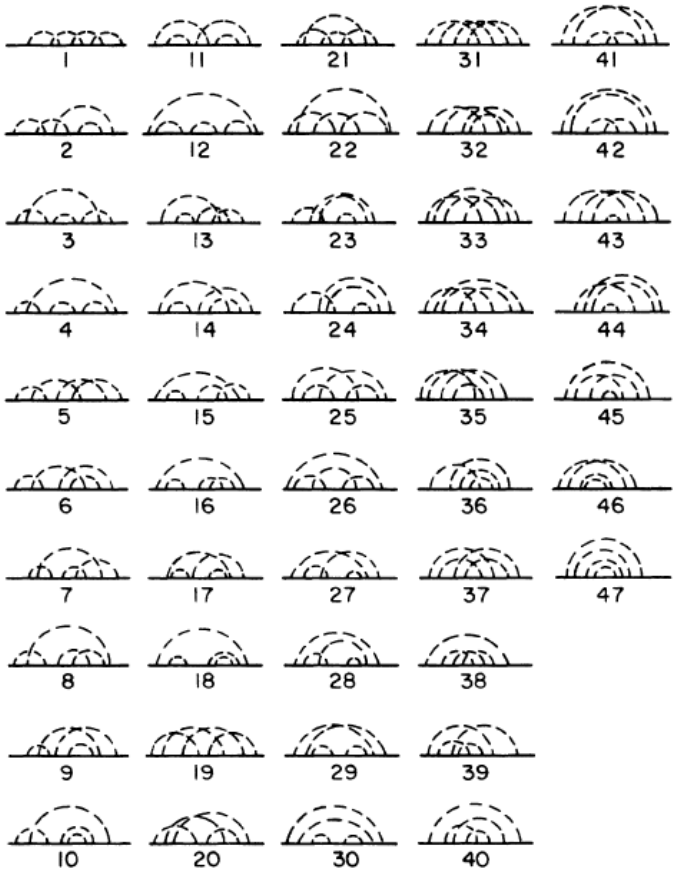
$$a_e = F(\alpha)$$

$$\alpha \equiv \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{1}{137.035\ 999\ 074(44)}$$

- Mag. Moment confirmed by perturbation theory

$$a_e = 0.5\left(\frac{\alpha}{\pi}\right) - 0.328\ 478\ 965\ 579\left(\frac{\alpha}{\pi}\right)^2 + 1.181\ 241\ 456\ 587\left(\frac{\alpha}{\pi}\right)^3 - 1.509\ 8(384)\left(\frac{\alpha}{\pi}\right)^4$$

What if $\alpha = 100$?



non-perturbative methods so far

- Classical: (ex) monopole, instanton
- Trivial theories: (ex) g=0 or infinity
- Trivial quantities: (ex) BPS
- Qualitative: (ex) holography
- Non-dynamical: (ex) localization

Real Non-perturbative theory?

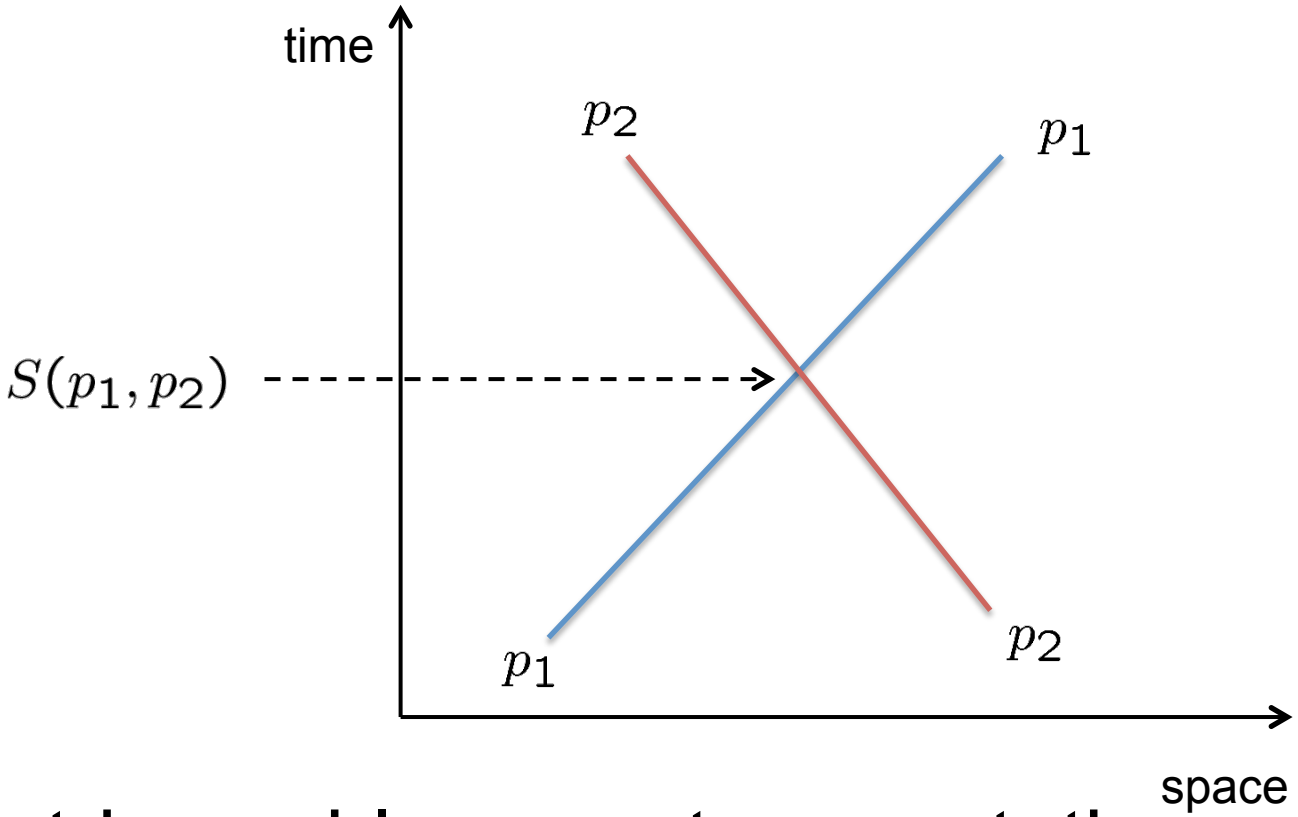
- Very difficult
- Only way is EXACT SOLUTION
- INTEGRABILITY

SYMMETRY → exact solution

S-matrix approach

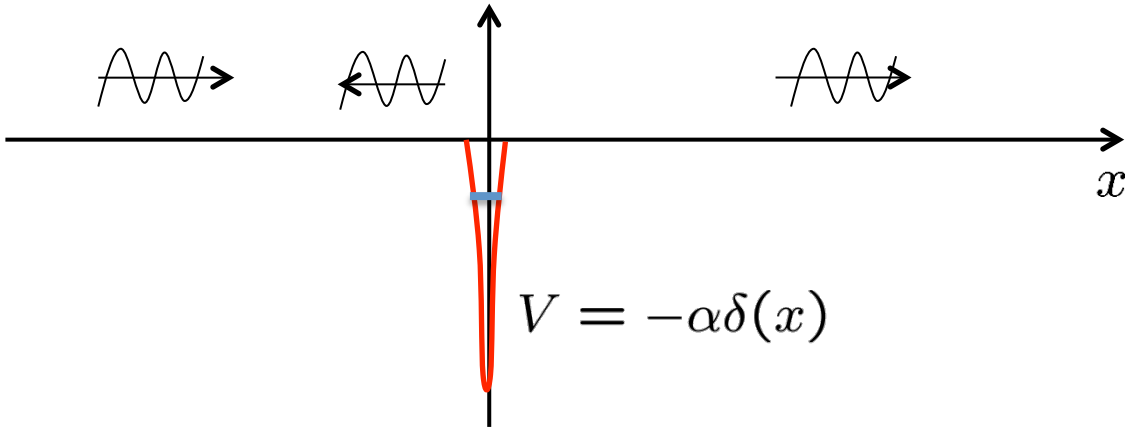
S-matrix bootstrap

- Infinite charges→ completely elastic scattering



- S-matrix enables exact computations of certain physical quantities

- Pole of S → bound state
- (ex) delta-function potential in QM



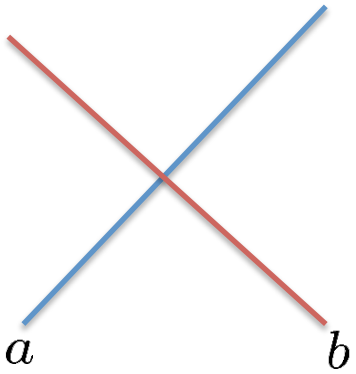
– Bound state energy: $E = -\frac{m\alpha^2}{2\hbar^2}$ Griffiths pp.73-74

– Scattering amplitudes: (E>0)

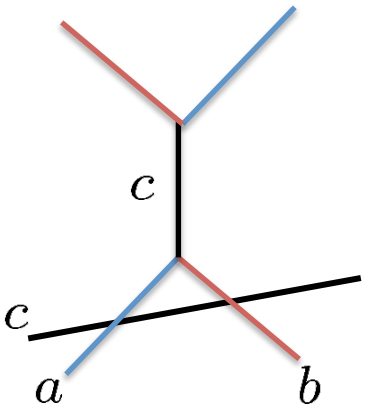
$$r(p) = \frac{i\beta}{1-i\beta}, \quad t(p) = \frac{i\beta}{1-i\beta}, \quad \beta \equiv \frac{m\alpha}{\hbar p}$$

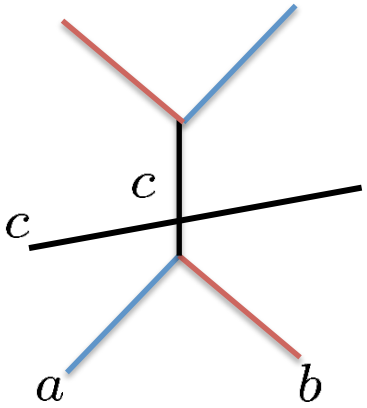
Pole at $p = i\frac{m\alpha}{\hbar}$ -----> $E = -\frac{m\alpha^2}{2\hbar^2}$

S-matrix Bootstrap



S_{ab}





$$S_{ca}S_{cb} \qquad = \qquad S_{cc}$$

(ex) 2d Ising model with magnetic field

- S-matrices by Zamolodchikov (1988)

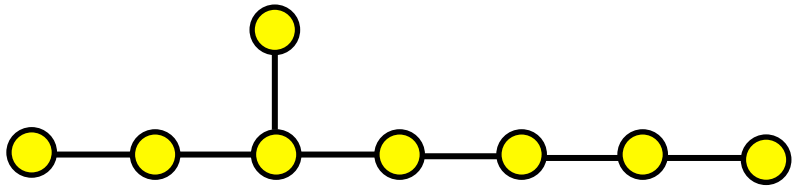
- Mass spectrum from poles of S-matrices

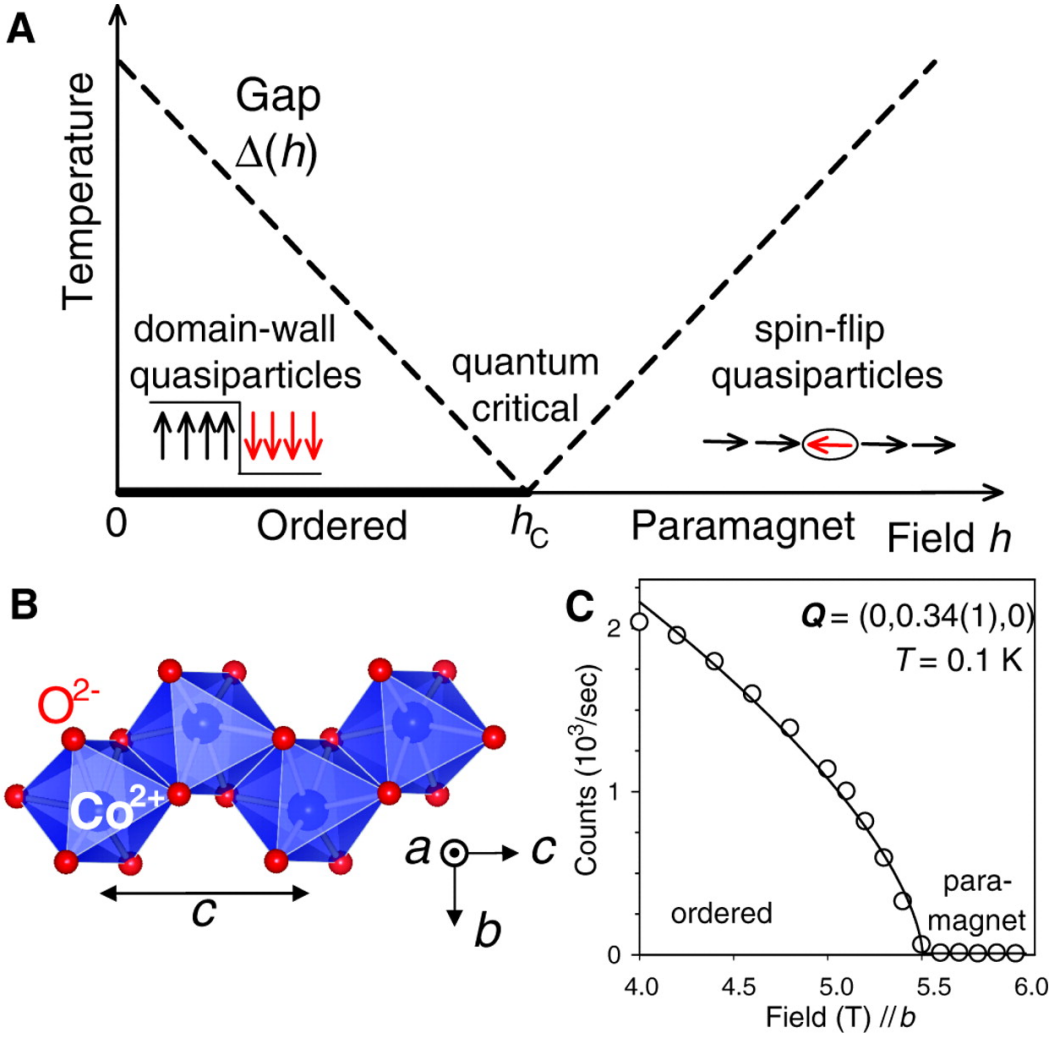
$$m_1 = (4.40490857\ldots)|B|^{5/13},$$
$$m_2 = 2m_1 \cos\frac{\pi}{5}, \dashrightarrow \frac{\sqrt{5}+1}{2}$$
$$m_3 = 2m_1 \cos\frac{\pi}{30},$$
$$m_4 = 4m_1 \cos\frac{\pi}{5}\cos\frac{7\pi}{30},$$
$$m_5 = 4m_1 \cos\frac{\pi}{5}\cos\frac{2\pi}{15},$$
$$m_6 = 4m_1 \cos\frac{\pi}{5}\cos\frac{\pi}{30},$$
$$m_7 = 8m_1 \cos\frac{\pi}{5}\cos\frac{\pi}{5}\cos\frac{7\pi}{30},$$
$$m_8 = 8m_1 \cos\frac{\pi}{5}\cos\frac{\pi}{5}\cos\frac{2\pi}{15}$$

Golden ratio

Perron-Frobenius vector of Cartan matrix

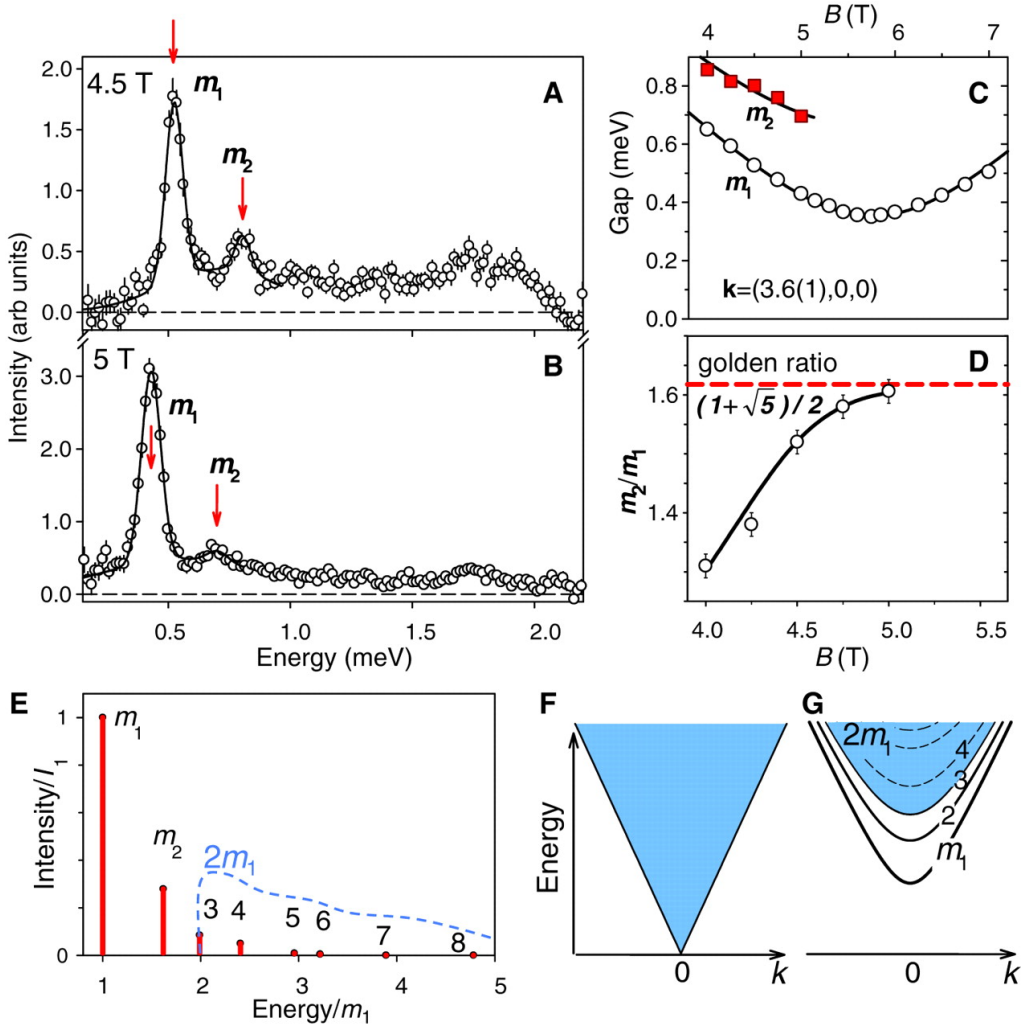
E_8 Lie algebra





R Coldea et al. Science 2010;327:177-180

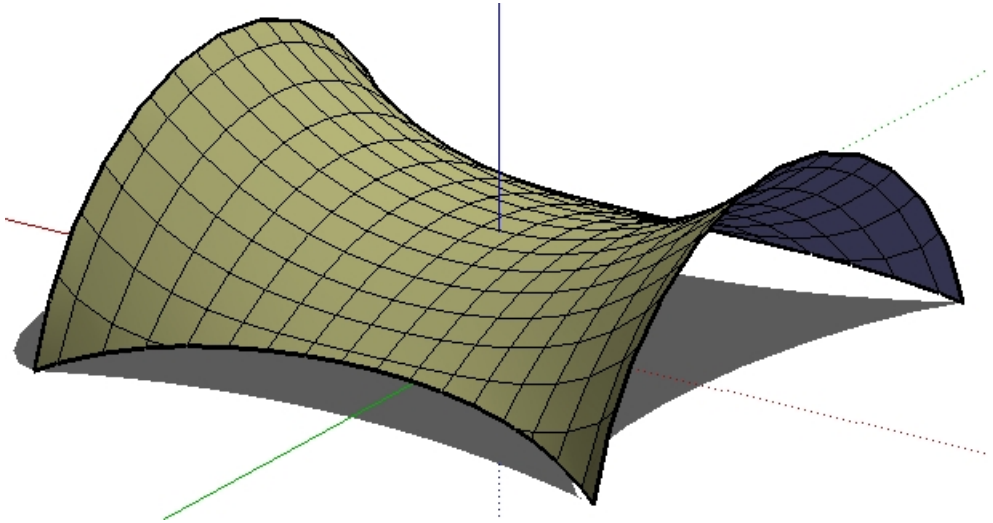
(A and B) Energy scans at the zone center at 4.5 and 5 T observing two peaks, m1 and m2, at low energies.



R Coldea et al. Science 2010;327:177-180

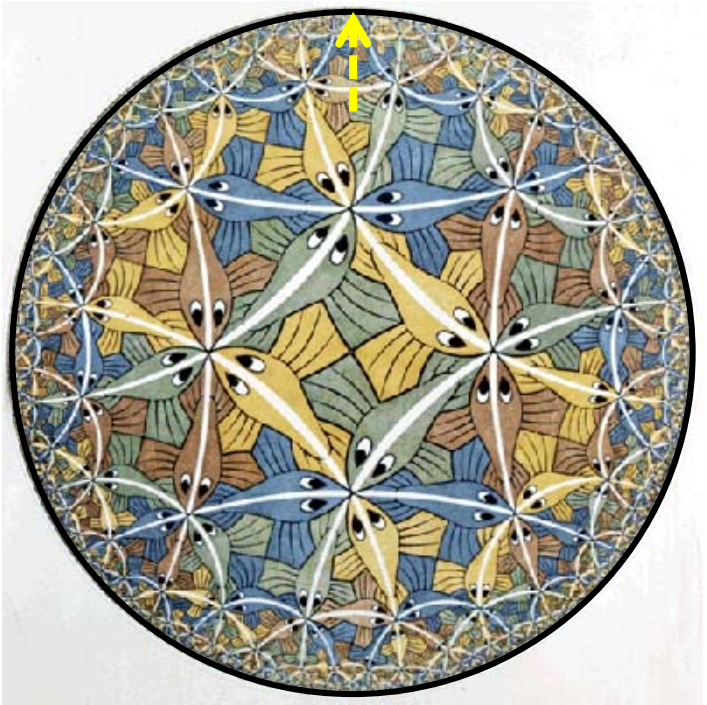
AdS / CFT duality

Space with negative curvature



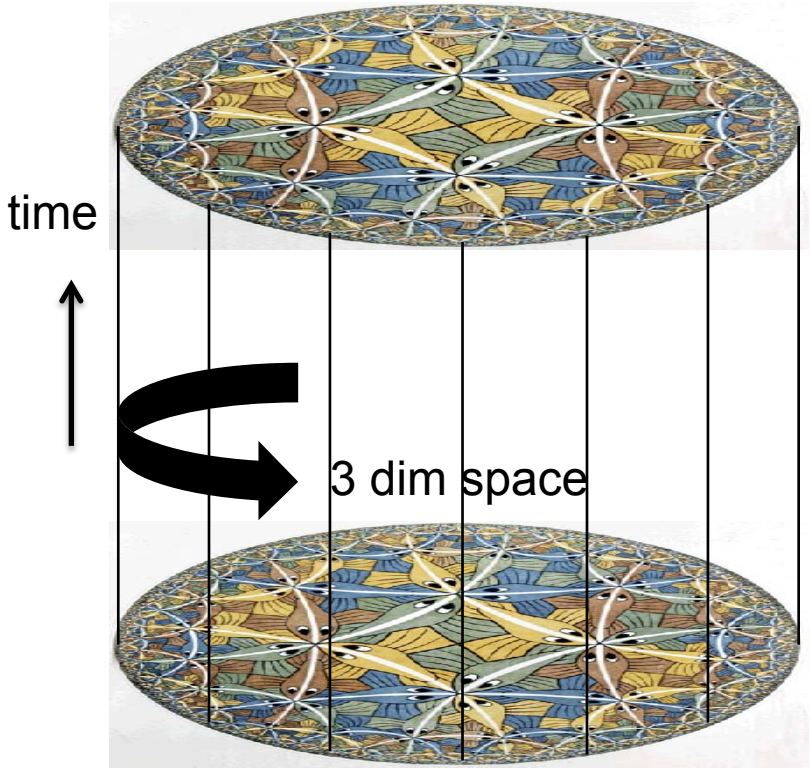
∞

- Anti-de Sitter (AdS) space



AdS space

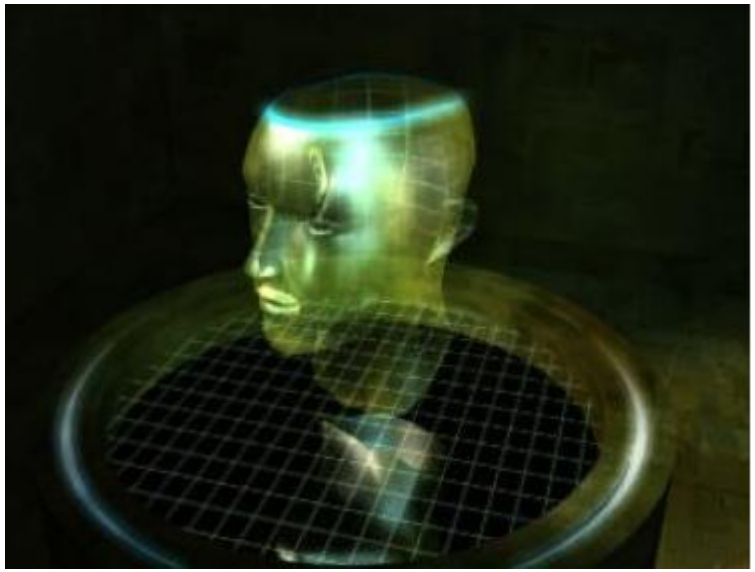
- Inside: 5 dim AdS_5 space



- Boundary: 4 dim space-time

Holography principle

- Hologram

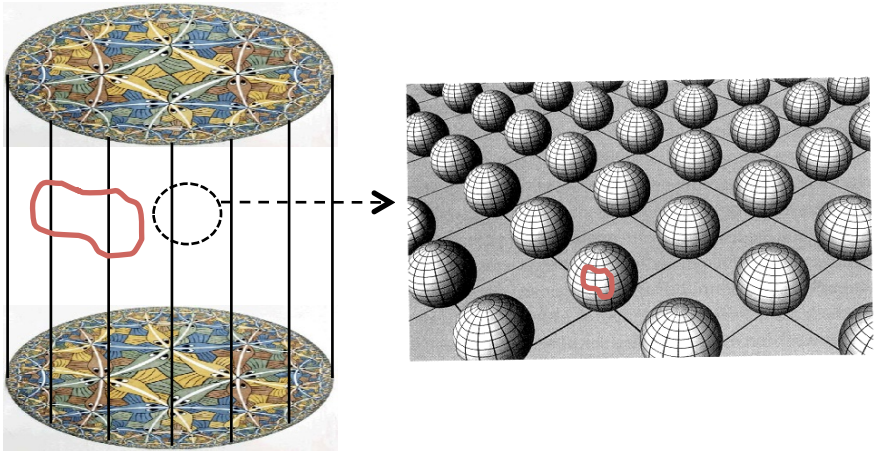


- Holography principle
  - String theory on AdS space = gauge theory on the boundary space-time
  - Quantum gravity effect ←→ gauge interaction

$$\frac{1}{\lambda} \qquad \lambda$$

AdS/CFT conjecture (Maldacena)

- String theory on $AdS_5 \times S^5$



- 4 dim super Yang-Mills theory

Gauge theory

- Electromagnetism (Abelian gauge theory)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- Yang-Mills (non-Abelian gauge) theory:

$$A_\mu = N_c \times N_c \text{ Matrix}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] = \frac{1}{2}\sum_a \left(\mathbf{E}_a{}^2 - \mathbf{B}_a{}^2\right)$$

Supersymmetric Yang-Mills theory

- Supersymmetry
  - Scalar boson, fermion, gauge boson

  - N=4 SYM

$$(A_\mu, \chi^a_\alpha, \Phi^j), \quad a = 1, \ldots, 4; \; j = 1, \ldots, 6$$

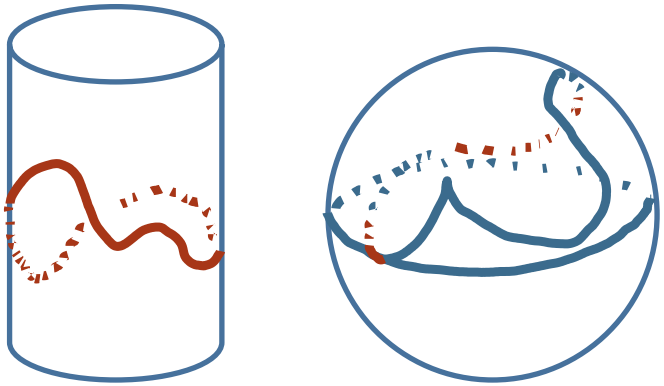
$$S = \frac{\mathrm{Tr}}{g^2_{\mathrm{YM}}} \int d^4x \left\{ -\frac{1}{4} F^2_{\mu\nu} + (D_\mu \Phi^a)^2 + \left[\Phi^a, \Phi^b\right]^2 + \bar{\chi} \not{D} \chi - i \bar{\chi} \Gamma_a [\Phi^a, \chi] \right\}$$

  - RG fixed point: scale invariance

$$\beta \equiv \mu \frac{\partial g_{\mathrm{YM}}}{\partial \mu} = -\frac{g^3_{\mathrm{YM}}}{16\pi^2} \left(\frac{11}{3} N_c - \frac{1}{6} \sum_i^{6N_c} C_i - \frac{1}{3} \sum_j^{8N_c} \tilde{C}_j \right) = 0$$

AdS/CFT duality

Energy of string = Dimension of SYM



$$\mathcal{O}(x) = \mathrm{Tr}\left[XYZF_{\mu\nu}\chi^{\alpha}(D_{\mu}Y)\ldots\right]$$

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{1}{|x|^{2\Delta}}$$

Need non-perturbation

$$\lambda = g^2_{\mathrm{YM}}N_c \equiv g^2$$

Symmetry

S-matrix

Physical Quantities

S-matrix

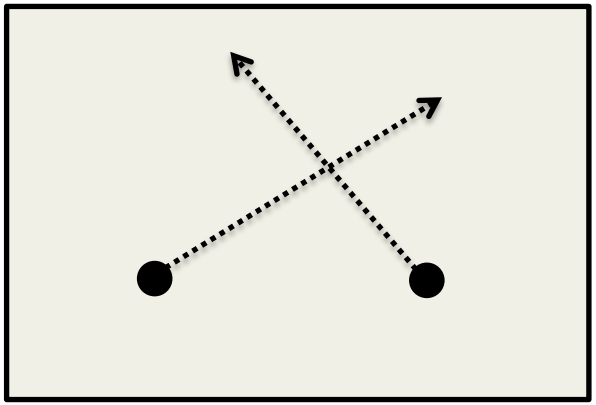
- SYM : scattering of fields on the spin chain

$$\overrightarrow{p_1} \qquad \overleftarrow{p_2}$$

$$\mathrm{Tr}\,[\cdots ZZZZZZZZZ\chi_1 ZZZZZZ\cdots ZZZZZZ\chi_2 ZZZZZZZZZZ\cdots]$$

$$\{X, \overline{X}, Y, \overline{Y}, \chi^{\alpha}, D_{\mu}\}$$

- String side : scattering on the world sheet

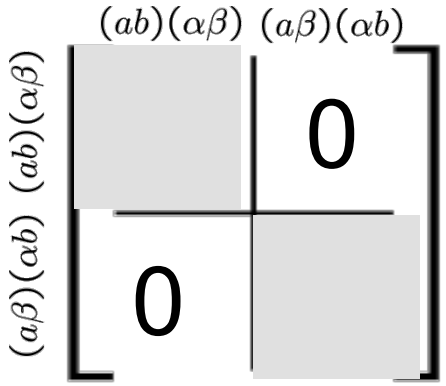


- Symmetry : 4d conformal group psu(2,2|4)

- From symmetry to S-matrix

$$\left[\mathbf{S}(p_1,p_2),\left(\begin{array}{c|c}\mathbb{L}_a{}^b & \mathbb{Q}_\alpha{}^b \\ \hline \mathbb{Q}_a^{\dagger\beta} & \mathbb{R}_\alpha{}^\beta\end{array}\right)\right]=0$$

- S : 16 x 16 matrix



$$
\begin{aligned}
S^{aa}_{aa} &= A\,, \quad S^{\alpha\alpha}_{\alpha\alpha} = D\,, \\
S^{ab}_{ab} &= \frac{1}{2}(A-B)\,, \quad S^{ba}_{ab} = \frac{1}{2}(A+B)\,, \\
S^{\alpha\beta}_{\alpha\beta} &= \frac{1}{2}(D-E)\,, \quad S^{\beta\alpha}_{\alpha\beta} = \frac{1}{2}(D+E)\,, \\
S^{\alpha\beta}_{ab} &= -\frac{1}{2}\epsilon_{ab}\epsilon^{\alpha\beta}\,C\,, \quad S^{ab}_{\alpha\beta} = -\frac{1}{2}\epsilon^{ab}\epsilon_{\alpha\beta}\,F\,, \\
S^{a\alpha}_{a\alpha} &= G\,, \quad S^{\alpha a}_{a\alpha} = H\,, \quad S^{a\alpha}_{\alpha a} = K\,, \quad S^{\alpha a}_{\alpha a} = L
\end{aligned}
$$

$$
\begin{aligned}
A &= S_0 \frac{x_2^- - x_1^+}{x_2^+ - x_1^-}\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\,, \\
B &= -S_0\left[\frac{x_2^- - x_1^+}{x_2^+ - x_1^-} + 2\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\right]\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\,, \\
C &= S_0 \frac{2ix_1^- x_2^-(x_1^+ - x_2^+)\eta_1\eta_2}{x_1^+ x_2^+(x_1^- - x_2^+)(1 - x_1^- x_2^-)}\,, \qquad D = -S_0\,, \\
E &= S_0\left[1 - 2\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- + x_2^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\right]\,, \\
F &= S_0 \frac{2i(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- - x_2^+)(1 - x_1^- x_2^-)\tilde\eta_1\tilde\eta_2}\,, \\
G &= S_0 \frac{(x_2^- - x_1^-)}{(x_2^+ - x_1^-)}\frac{\eta_1}{\tilde\eta_1}\,, \qquad H = S_0 \frac{(x_2^+ - x_2^-)}{(x_1^- - x_2^+)}\frac{\eta_1}{\tilde\eta_2}\,, \\
K &= S_0 \frac{(x_1^+ - x_1^-)}{(x_1^- - x_2^+)}\frac{\eta_2}{\tilde\eta_1}\,, \qquad L = S_0 \frac{(x_1^+ - x_2^+)}{(x_1^- - x_2^+)}\frac{\eta_2}{\tilde\eta_2}
\end{aligned}
$$

$$
\eta_1 = \eta(p_1)e^{ip_2/2}\,, \quad \eta_2 = \eta(p_2)\,, \quad \tilde\eta_1 = \eta(p_1)\,, \quad \tilde\eta_2 = \eta(p_2)e^{ip_1/2}
$$

beta-deformed SYM

$$V_\beta = \left|e^{i\pi\beta}ZX - e^{-i\pi\beta}XZ\right|^2 + \left|e^{i\pi\beta}XY - e^{-i\pi\beta}YX\right|^2 + \left|e^{i\pi\beta}YZ - e^{-i\pi\beta}ZY\right|^2$$

- N=1 super-CFT
- Dual to string theory on Lunin-Maldacena background $AdS_5 \times S^5_\beta$

S-matrix

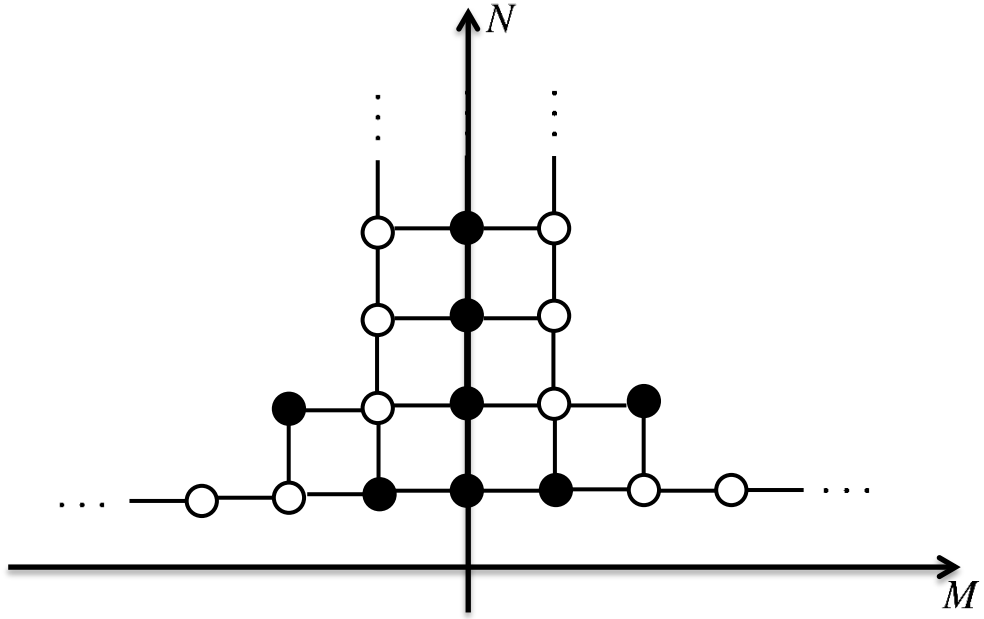
[CA-Bajnok-Bombardelli-Nepomechie (2010a)]

- Drinfeld-twist by a constant matrix

$$S_\beta(p_1, p_2) = F_\beta \cdot S(p_1, p_2) \cdot F_\beta$$

Y-system

$$\ln Y_{N,M} = s\star\Big[\ln(1+Y_{N,M+1}) + \ln(1+Y_{N,M-1})\Big] - s\star\Big[\ln(1+Y^{-1}_{N+1,M}) + \ln(1+Y^{-1}_{N-1,M})\Big]$$

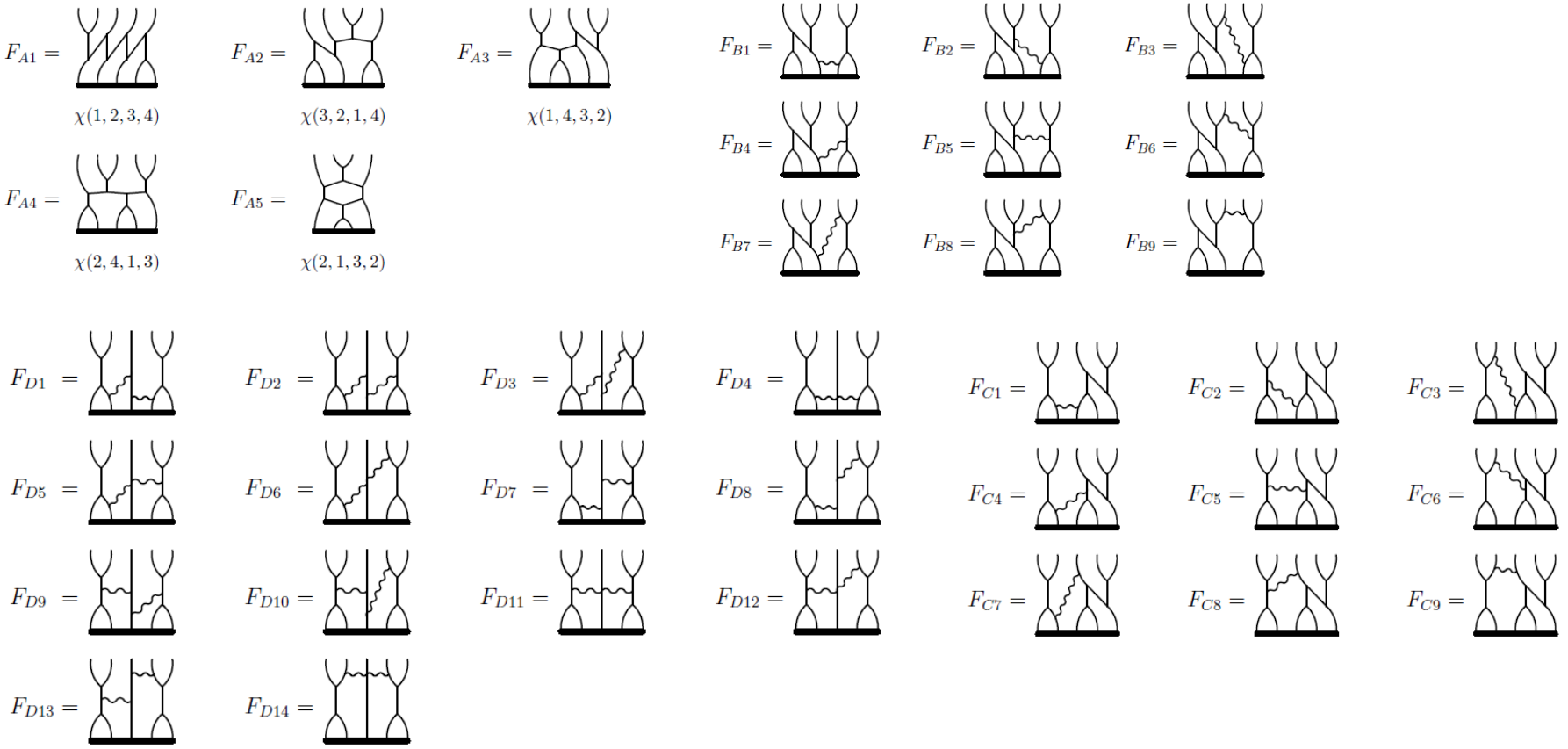


$$\Delta = -\sum_{N=1}^{\infty}\int\frac{dp}{2\pi}\log(1+Y_{N,0}) = \Delta(g)$$

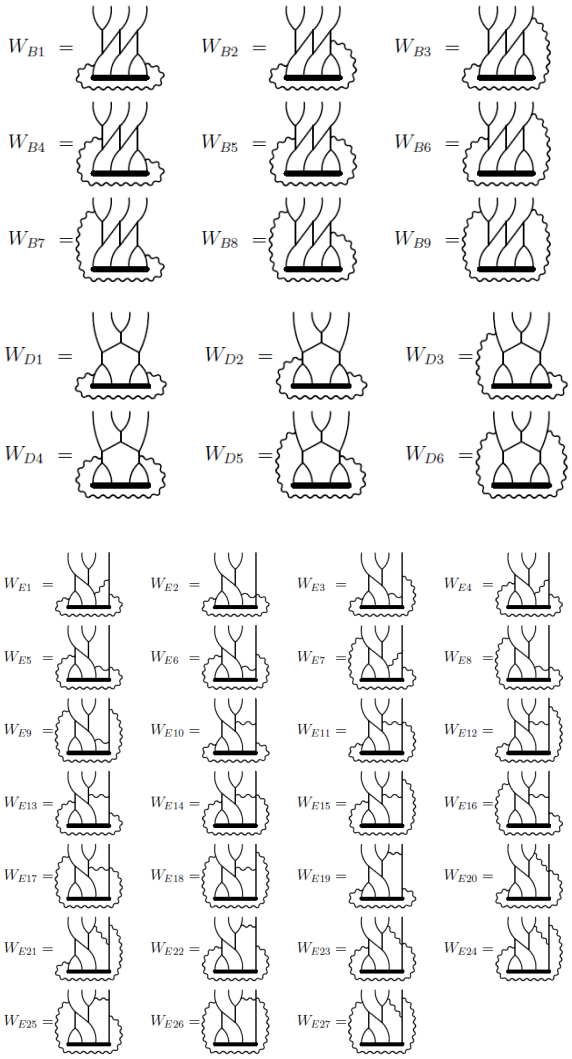
4-Loop su(2) Konishi

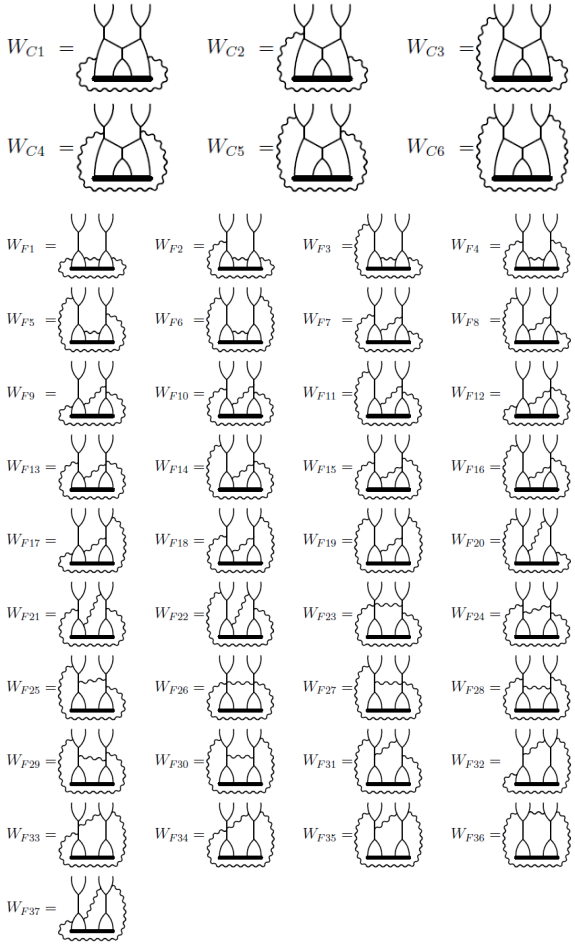
[Fiamberti, Santambrogio, Sieg, Zanon (2008)]

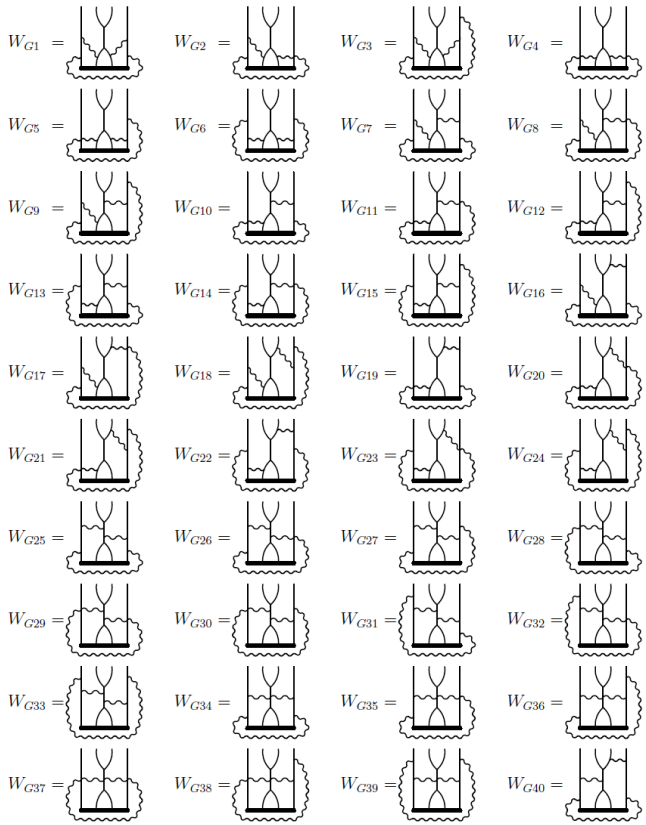
N=1 supergraphs



Wrapping diagrams







$$
\begin{aligned}
\gamma &= 4 + g^2\gamma_1 + g^4\gamma_2 + g^6\gamma_3 + g^8\gamma_4 + \cdots \\
\gamma_1 &= 6(1+\delta) \\
\gamma_2 &= -\frac{3}{\delta} - 15 - 21\delta - 9\delta^2 \\
\gamma_3 &= -\frac{3}{4\delta^3} + \frac{153}{4\delta} + 114 + \frac{495}{4}\delta + 54\delta^2 + \frac{27}{4}\delta^3 \\
\gamma_4 &= -\frac{3}{8\delta^5} + \frac{33}{2\delta^3} - \frac{1701}{4\delta} - 1230 \\
&- \frac{2427}{2}\delta - 180\delta^2 + 162\delta^4 + \frac{2997}{8}\delta^3 \\
&+ \left(-\frac{9}{\delta} + 297 + 702\delta + 234\delta^2 - 405\delta^3 - 243\delta^4\right)\zeta(3) \\
&- 360\,(1+\delta)^2\,\zeta(5)
\end{aligned}
$$

$$
\delta \equiv \frac{\sqrt{5 + 4\cos(4\pi\beta)}}{3}
$$

Results from Y-system

[CA-Bajnok-Bombardelli-Nepomechie (2010b)]

$$\Delta = -\sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{4^L g^{2L}}{(Q^2+q^2)^L} \sum_{j,j'} (-1)^{F_{(jj')}} \left[\mathcal{S}^{(Q1)}(q,p) \mathcal{S}^{(Q1)}(q,-p) \right]^{(jj')(11)}_{(jj')(11)}$$

After residue integrals

$$\sum_{Q=1}^{\infty} \left[\frac{f_1}{Q^5} + \frac{f_2}{Q^3} + f_3(Q) \right]$$

$$f_1 = -\frac{2560(1+2u_1^2+2u_2^2)^2}{(4u_1^2+1)^2(4u_2^2+1)^2}$$

$$f_2 = \frac{\text{num}}{(4u_1^2+1)^4(4u_2^2+1)^4}$$

$$\begin{aligned}
\text{num} &= 2048\Big(-1+5u_1^2+48u_1^4+96u_1^6-2u_1u_2-16u_1^3u_2-32u_1^5u_2+5u_2^2+224u_1^2u_2^2 \\
&+ 1024u_1^4u_2^2+1536u_1^6u_2^2+768u_1^8u_2^2-16u_1u_2^3-128u_1^3u_2^3+64u_1^8 \\
&- 256u_1^5u_2^3+48u_2^4+1024u_1^2u_2^4+3200u_1^4u_2^4+2560u_1^6u_2^4-32u_1u_2^5 \\
&- 256u_1^3u_2^5-512u_1^5u_2^5+96u_2^6+1536u_1^2u_2^6+2560u_1^4u_2^6+64u_2^8+768u_1^2u_2^8\Big)
\end{aligned}$$

f_3 = 1775 terms

$$u_{1,2} = \frac{(1-3\,\delta)^2}{2\sqrt{-1+9\,\delta^2}\left(3\sqrt{1-\delta^2} \pm 2\sqrt{\frac{3+3\delta}{1+3\delta}}\right)}$$

$$
\begin{aligned}
-(&512\,i\,(-5\,i+156\,i\,Q^2-2120\,i\,Q^4+16512\,i\,Q^6-77952\,i\,Q^8+216064\,i\,Q^{10}-317440\,i\,Q^{12}+196608\,i\,Q^{14}-32768\,i\,Q^{16}-160\,Q\,u1+4352\,Q^3\,u1-\\
&50432\,Q^5\,u1+321408\,Q^7\,u1-1178624\,Q^9\,u1+2396160\,Q^{11}\,u1-2490368\,Q^{13}\,u1+1277952\,Q^{15}\,u1-262144\,Q^{17}\,u1-140\,i\,u1^2+\\
&6036\,i\,Q^2\,u1^2-97280\,i\,Q^4\,u1^2+813504\,i\,Q^6\,u1^2-3904512\,i\,Q^8\,u1^2+10729472\,i\,Q^{10}\,u1^2-16007168\,i\,Q^{12}\,u1^2+12238848\,i\,Q^{14}\,u1^2-\\
&4456448\,i\,Q^{16}\,u1^2+786432\,i\,Q^{18}\,u1^2-3840\,Q\,u1^3+109184\,Q^3\,u1^3-1286656\,Q^5\,u1^3+8186112\,Q^7\,u1^3-29806080\,Q^9\,u1^3+60510208\,Q^{11}\,u1^3-\\
&65617920\,Q^{13}\,u1^3+38076416\,Q^{15}\,u1^3-11403264\,Q^{17}\,u1^3+1048576\,Q^{19}\,u1^3-1700\,i\,u1^4+85248\,i\,Q^2\,u1^4-1436928\,i\,Q^4\,u1^4+\\
&12094720\,i\,Q^6\,u1^4-57501696\,i\,Q^8\,u1^4+155193344\,i\,Q^{10}\,u1^4-229015552\,i\,Q^{12}\,u1^4+182779904\,i\,Q^{14}\,u1^4-75235328\,i\,Q^{16}\,u1^4+\\
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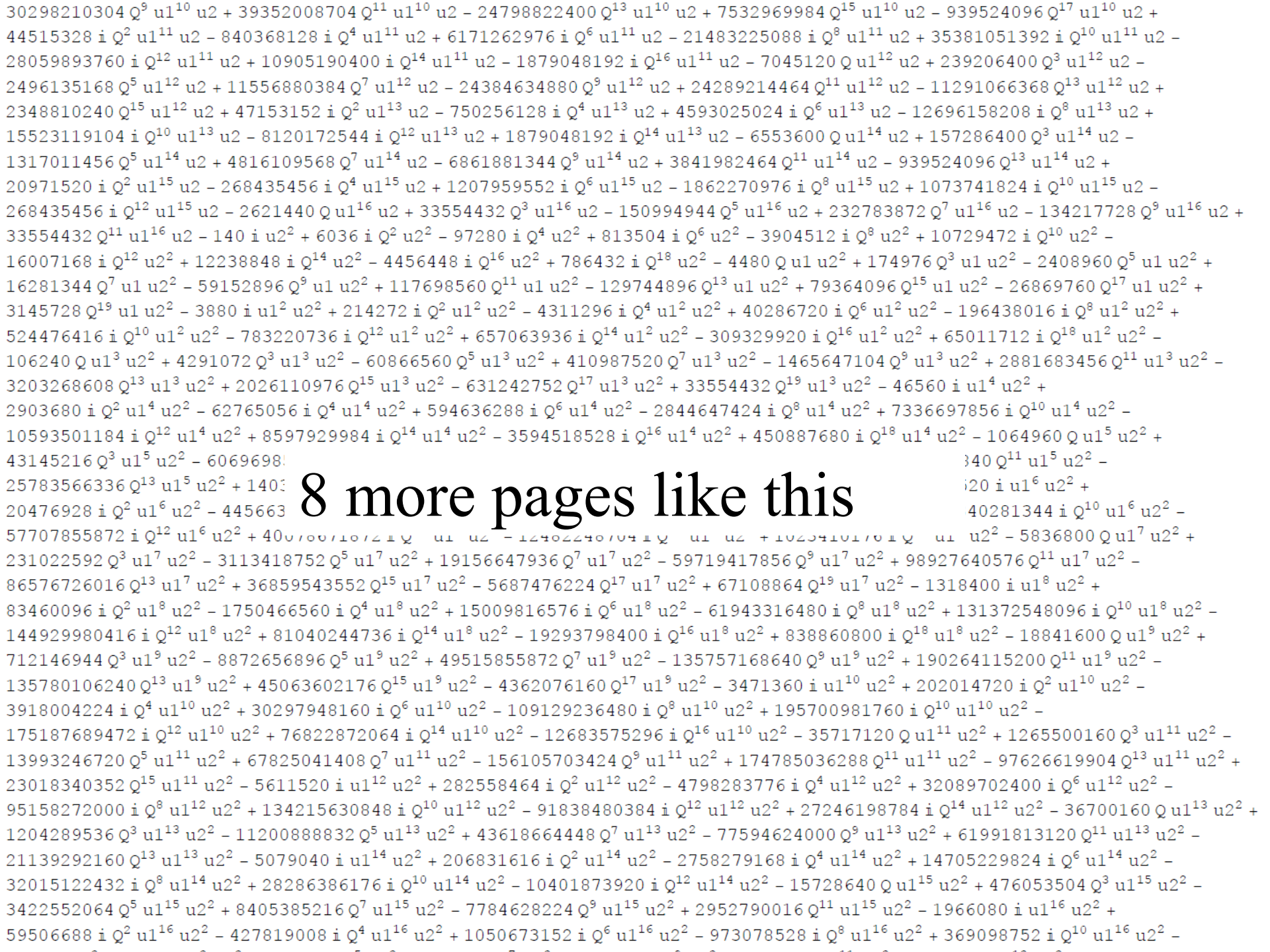
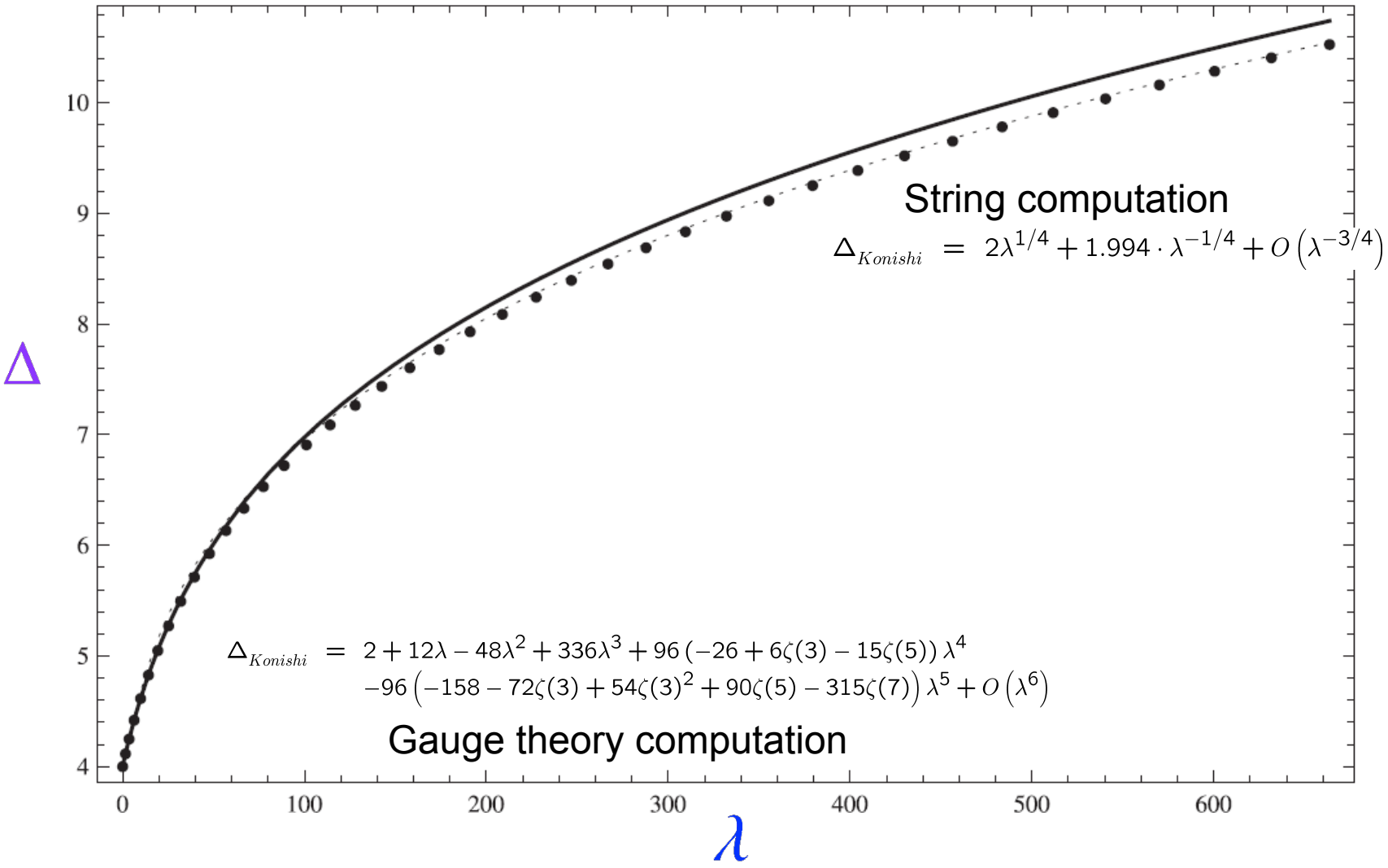
\end{aligned}
$$


8 more pages like this

- But the sum can be done by maple
- At four loop it match with Feynman diagrams

- No five loop yet
- But S-matrix gives EXACT formula!

- Exact solution



Conclusion

perturbation

Non-perturbation

Rigorous, analytic understanding based on integrability is at the core of non-perturbative phenomena

Thank you