Beyond Perturbation Theory

Integrability:
toward Non-perturbative physics

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“Non” in Physics

Non-linear

Non-Fermi

Non-Gaussian

Non-perturbative

…

Perturbation , 攝動

- Approximation
- Exact solutions for a simpler problem first
- Small additional term based on the exact sols.

(ex) Quantum Mechanics

- Energy level corrections by perturbation

$$H|\Psi_n\rangle = E_n|\Psi_n\rangle$$

$$H = H_0 + \lambda H', \qquad H_0|\Psi_n^0\rangle = E_n^0|\Psi_n^0\rangle$$

$$E_n = E_n^0 + \lambda\langle\Psi_n^0|H'|\Psi_n^0\rangle + \lambda^2 \sum_{m\neq n} \frac{|\langle\Psi_n^0|H'|\Psi_n^0\rangle|^2}{E_n^0 - E_m^0} + \dots$$

- Valid only when λ is small enough

Taylor expansion

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

Valid if $x \ll 1$

Invalid if $x > 1$

Need many terms if $x < 1$

Successes

- Perturbations are applied to many systems

- Successes of physics in the 20 century are all based on perturbation
- Feynman diagram

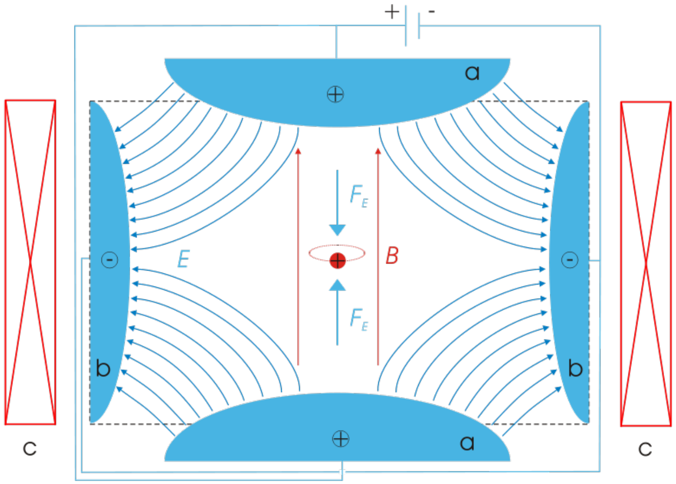
$$E_n = E_n^0 + \lambda \langle \Psi_n^0 | H' | \Psi_n^0 \rangle + \lambda^2 \sum_{m \neq n} \frac{|\langle \Psi_n^0 | H' | \Psi_n^0 \rangle|^2}{E_n^0 - E_m^0} + \dots$$

- (ex) Quantum Electrodynamics

Magnetic moment of electron

- Lande g-factor $$\boldsymbol{\mu}_e = g\frac{(-e)}{2m_e}\boldsymbol{S}$$
  - Dirac Eq. $$g = 2$$
  - Exp.: $$a_e \equiv \frac{g-2}{2} = 0.001\ 159\ 652\ 180\ 85(76)$$

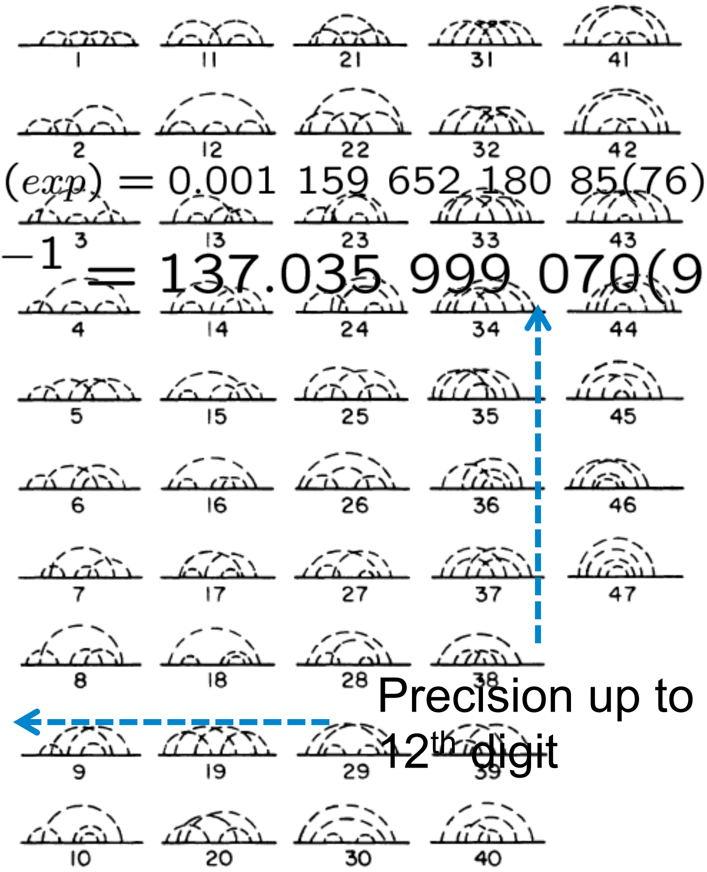
Penning Trap



the most accurately verified prediction in the history of physics

- QED perturbation theory

$$a_e = 0.5\left(\frac{\alpha}{\pi}\right) - 0.328\ 478\ 965\ 579\left(\frac{\alpha}{\pi}\right)^2 + 1.181\ 241\ 456\ 587\left(\frac{\alpha}{\pi}\right)^3 - 1.509\ 8(384)\left(\frac{\alpha}{\pi}\right)^4$$



$$a_e(exp) = 0.001\ 159\ 652\ 180\ 85(76)$$

$$\rightarrow \alpha^{-1} = 137.035\ 999\ 070(98)$$

Precision up to 12th digit

$$\alpha \equiv \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{1}{137.035\ 999\ 074(44)}$$

Measured by Qauntum Hall Effect

Fundamental Forces in Nature

- Electromagnetic Interaction
  - Condensed matter systems
  - Quantum Electrodynamics (QED)
- Weak Interaction
  - Standard model of elementary particle physics
  - Interactions are small
- Gravity
  - Even weaker than the weak interaction
- Strong interaction

Strong Force (QCD)

- Quarks inside proton interacts very weakly
  - Asymptotic freedom (2004 Nobel prize in physics)

when very close or high energy

$$g_s \to 0$$

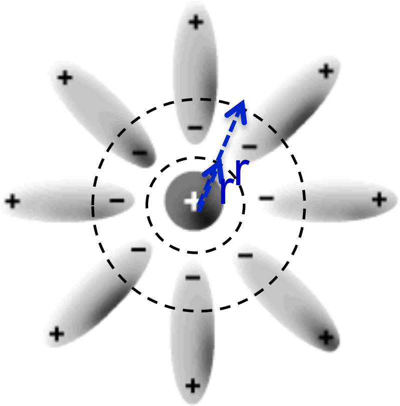
- Even strong interaction is perturbative!

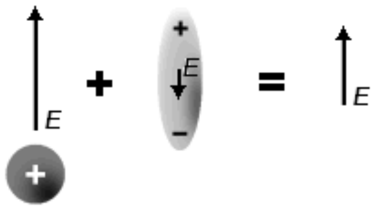
So, Perturbation is enough?

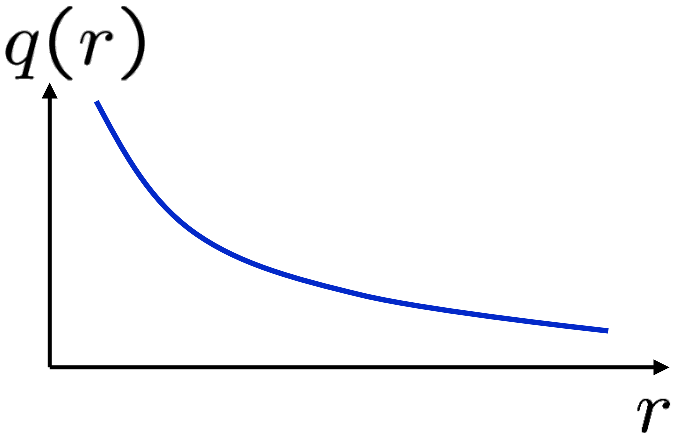
No!

There are many phenomena which are Non-perturbative

Renormalization

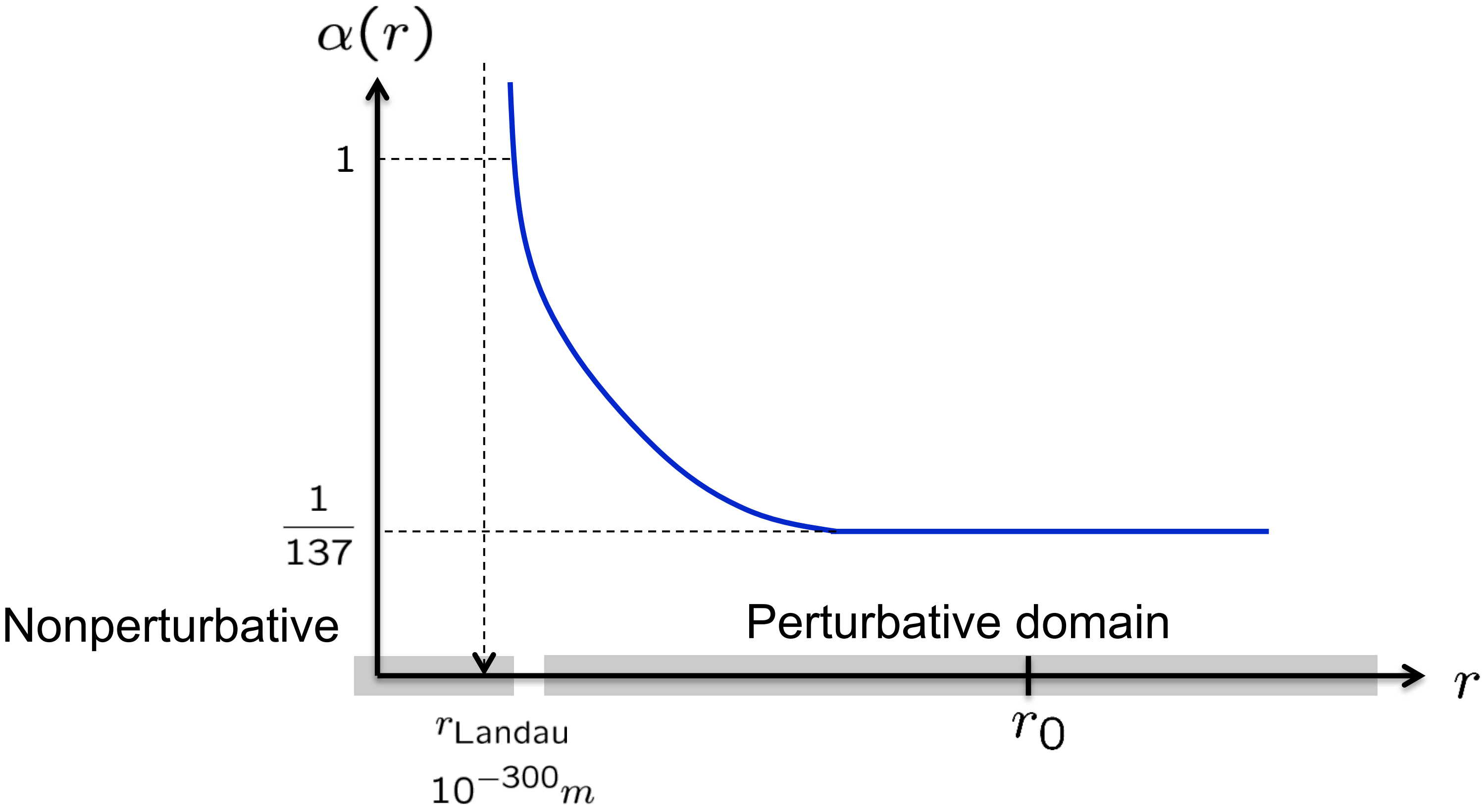






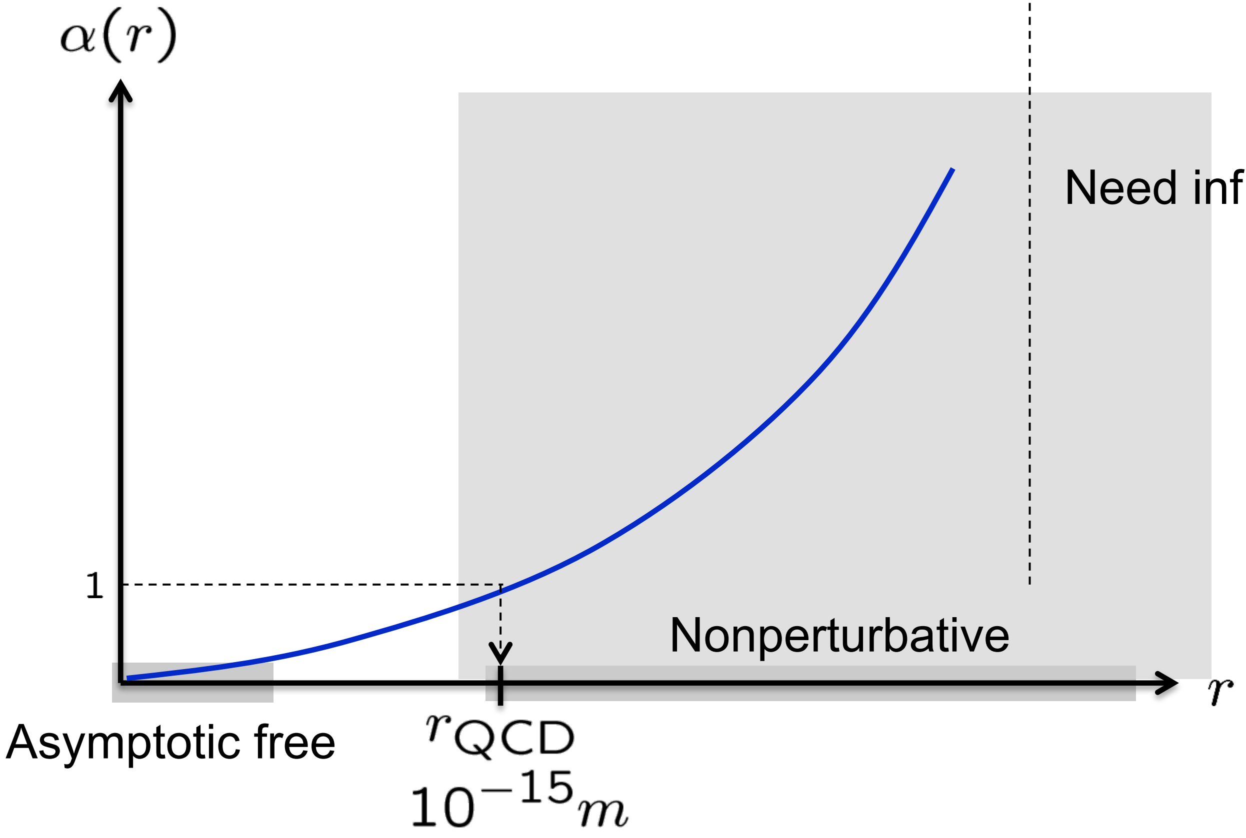
QED

$$\alpha(r) = \alpha - \frac{\alpha^2}{3\pi}\log\frac{r}{r_0} + \ldots \approx \frac{\alpha}{1+\frac{\alpha}{3\pi}\log\frac{r}{r_0}}$$



QCD

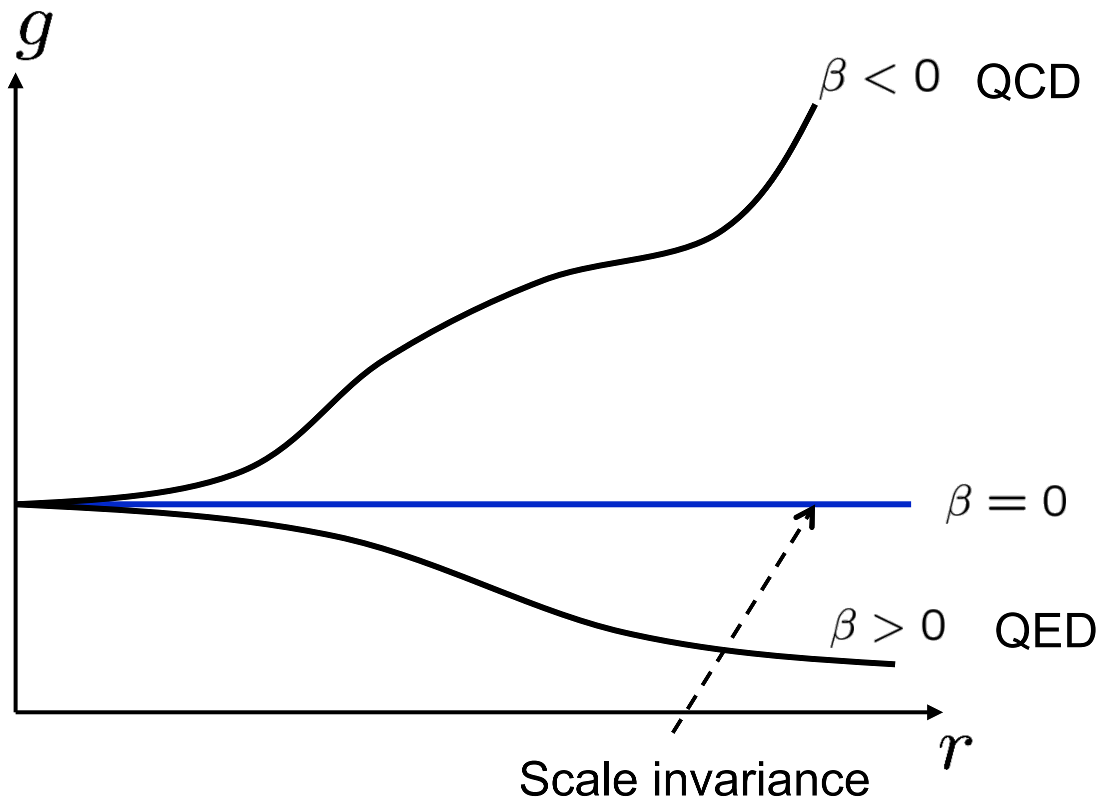
$$\alpha_s(r) = \alpha_s + \alpha_s^2 \left(\frac{11}{3}\underset{\overset{\uparrow}{3}}{N_c} - \frac{2}{3}\underset{\overset{\uparrow}{6}}{n_f}\right)\log\frac{r}{r_0} + \underset{\uparrow}{\ldots} \approx \xcancel{\frac{\alpha_s}{1-7\alpha_s\log\frac{r}{r_0}}}$$



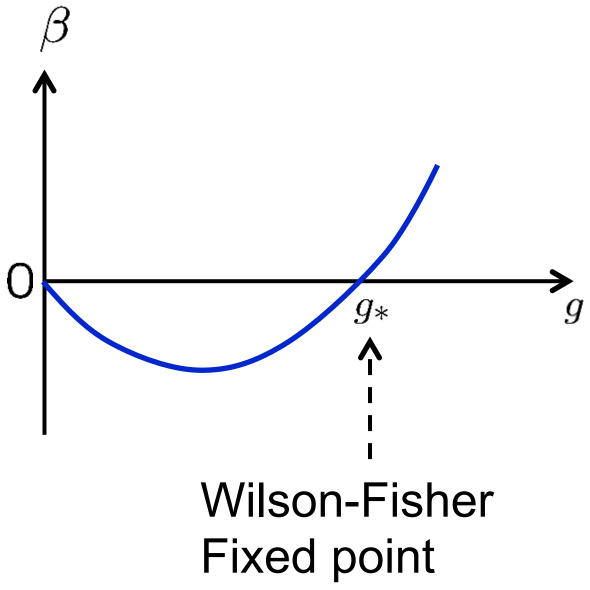
Need infinite terms

Renormalization group

$$\frac{d}{d \log r} g \equiv -\beta(g)$$



Scale invariance



Wilson-Fisher
Fixed point

Condensed matter system

- Landau: metals as fermi gas with perturbation (Fermi liquid)
- Many interesting phenomena: Luttinger liquid, Mott transitions, etc show non-Fermi liquid
- Strongly correlated electron systems

Non-perturbative methods so far

- Classical: (ex) monopole, instanton
- Trivial theories: (ex) g=0 or infinity
- Trivial quantities: (ex) BPS
- Qualitative: (ex) holography
- Non-dynamical: (ex) localization

Real non-perturbative method

- Only way is EXACT SOLUTION
- INTEGRABILITY

SYMMETRY → exact solution

Integrable models

- Special models with infinite conserved charges
- Possible only in one dimensional space
- Experimental technics can now realize these
- Applicable to even 3 dimensional space

Examples

1. Low dimensional condensed matter systems

2. Gravity / Gauge or AdS/CFT

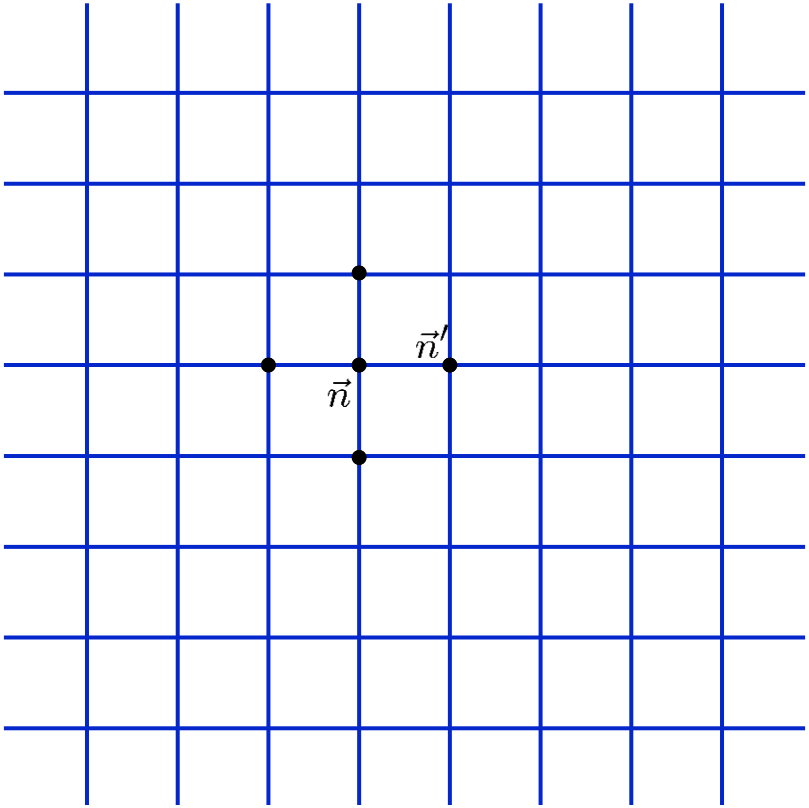
2D Ising model

$$E = -J \sum_{(\vec{n},\vec{n}')} s_{\vec{n}} s_{\vec{n}'} + B \sum_{\vec{n}} s_{\vec{n}}$$

Partition function

$$Z = \sum_{\{s\}} e^{-E/k_B T}$$

$s_{\vec{n}} = \pm 1$



RG Fixed point

$$\frac{J}{k_B T_c} = \frac{1}{2}\text{asinh}(1) = 0.4406\ldots$$

$B_c = 0$

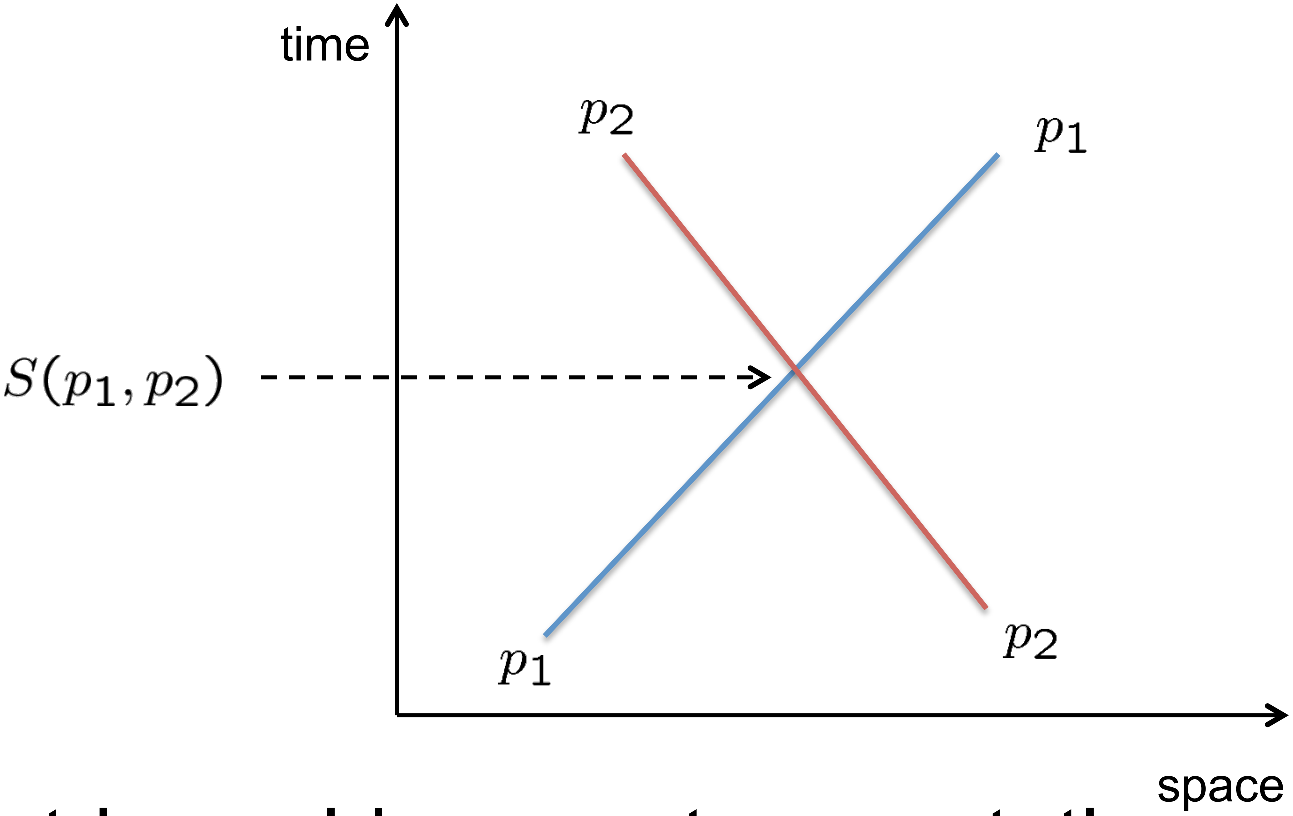
Onsager (1944)

$B = 0$

Can we solve for $B \neq 0$?

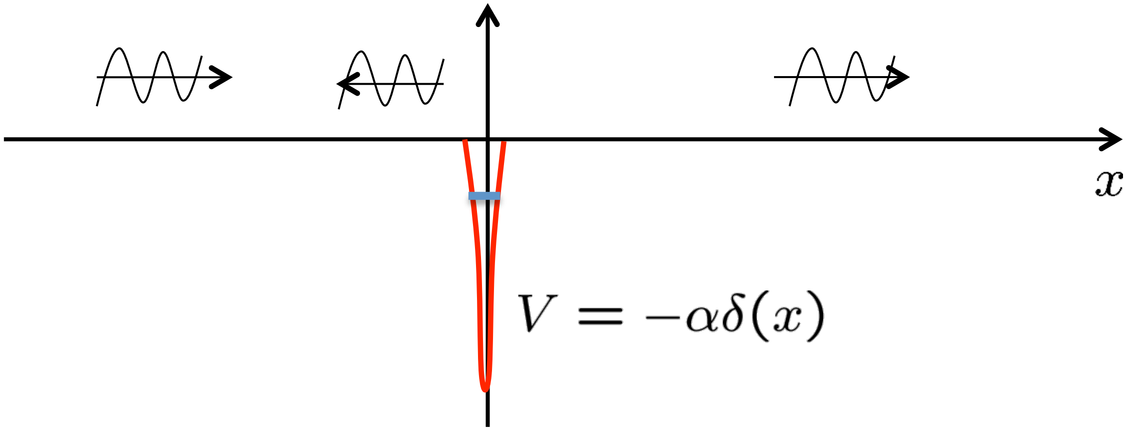
S-matrix

- Infinite charges→ completely elastic scattering



- S-matrix enables exact computations of certain physical quantities

- Pole of S → bound state
- (ex) delta-function potential in QM



- Bound state energy: $E = -\frac{m\alpha^2}{2\hbar^2}$

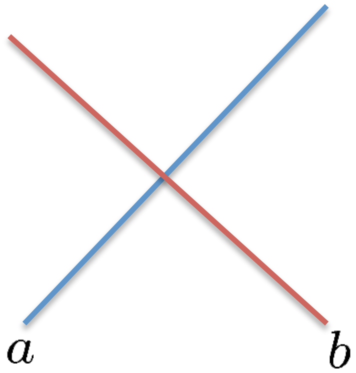
Griffiths pp.73-74

- Scattering amplitudes: (E>0)

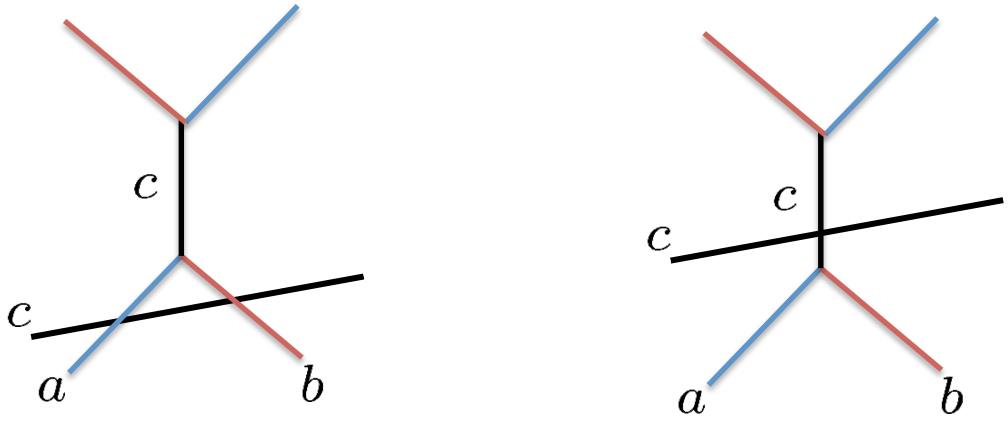
$$r(p) = \frac{i\beta}{1 - i\beta}, \quad t(p) = \frac{i\beta}{1 - i\beta}, \quad \beta \equiv \frac{m\alpha}{\hbar p}$$

Pole at $p = i\frac{m\alpha}{\hbar}$ -----> $E = -\frac{m\alpha^2}{2\hbar^2}$

Bootstrap



S_{ab}



$$S_{ca}S_{cb} = S_{cc}$$

Exact results

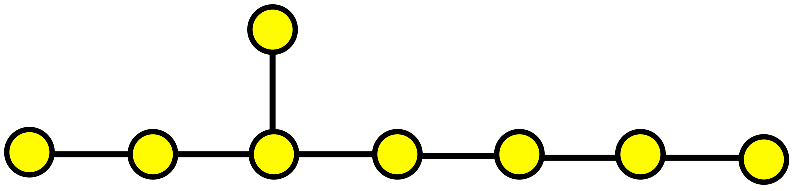
- S-matrices by Zamolodchikov (1988)
- Mass spectrum from poles of S-matrices

$$\left.\begin{aligned}
m_1 &= (4.40490857\ldots)|B|^{5/13}, \\
m_2 &= 2m_1 \cos\frac{\pi}{5}, \dashrightarrow \frac{\sqrt{5}+1}{2} \\
m_3 &= 2m_1 \cos\frac{\pi}{30}, \\
m_4 &= 4m_1 \cos\frac{\pi}{5}\cos\frac{7\pi}{30}, \\
m_5 &= 4m_1 \cos\frac{\pi}{5}\cos\frac{2\pi}{15}, \\
m_6 &= 4m_1 \cos\frac{\pi}{5}\cos\frac{\pi}{30}, \\
m_7 &= 8m_1 \cos\frac{\pi}{5}\cos\frac{\pi}{5}\cos\frac{7\pi}{30}, \\
m_8 &= 8m_1 \cos\frac{\pi}{5}\cos\frac{\pi}{5}\cos\frac{2\pi}{15}
\end{aligned}\right\}$$

Golden ratio

Perron-Frobenius vector
of Cartan matrix

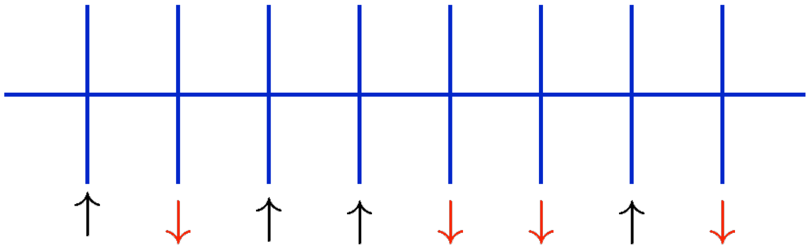
E_8 Lie algebra



Experimental evidence of E_8

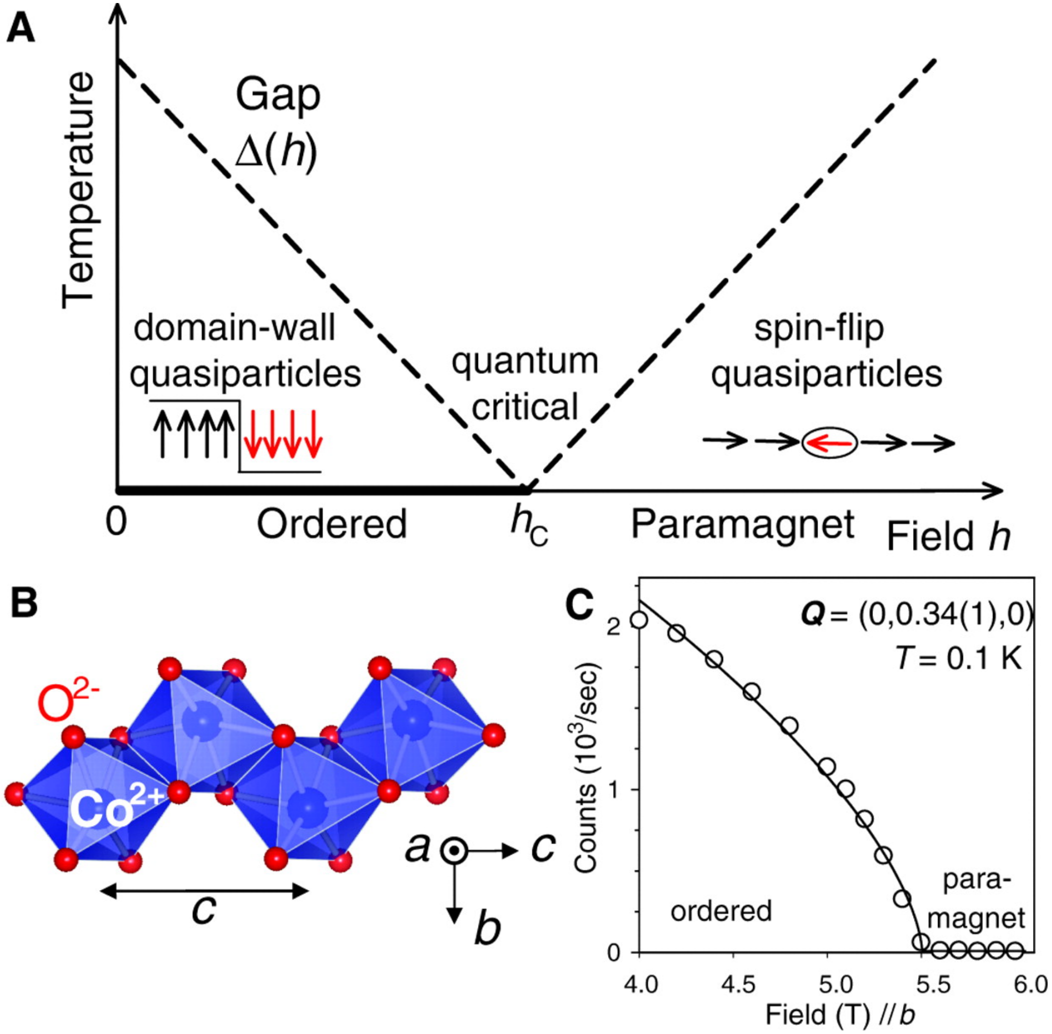
- 2D Ising model = 1D Ising spin chain

$$\hat{H} = -J \sum_{j=1} \left[\sigma^z_j \sigma^z_{j+1} + h \sigma^x_j \right] + B \sigma^z_j$$



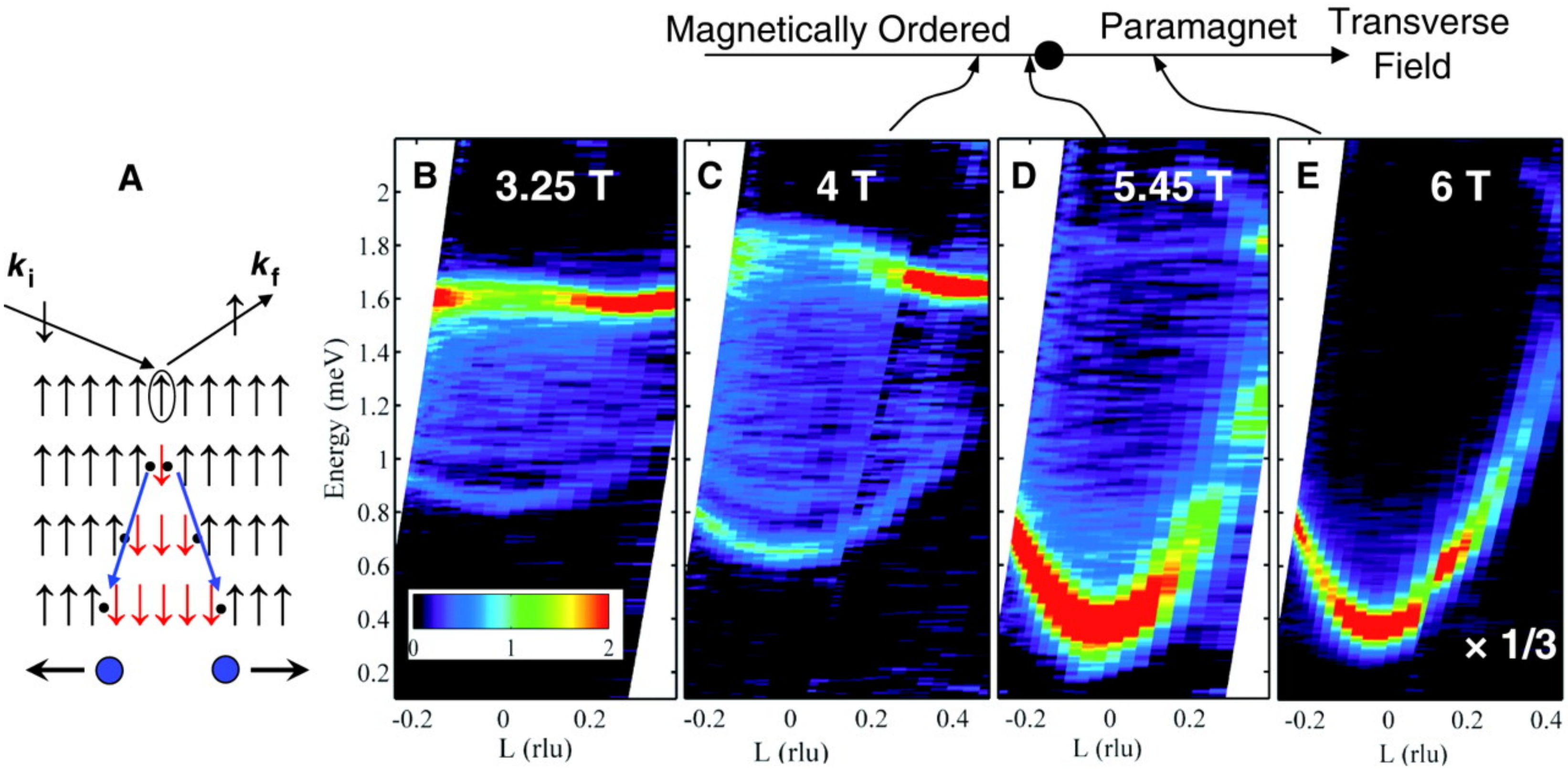
$$h = h_c = \frac{1}{2}$$ Critical Ising model

- Material: $CoNb_2O_6$



R Coldea et al. Science 2010;327:177-180

(A) Cartoon of a neutron spin-flip scattering that creates a pair of independently propagating kinks in a ferromagnetically ordered chain.



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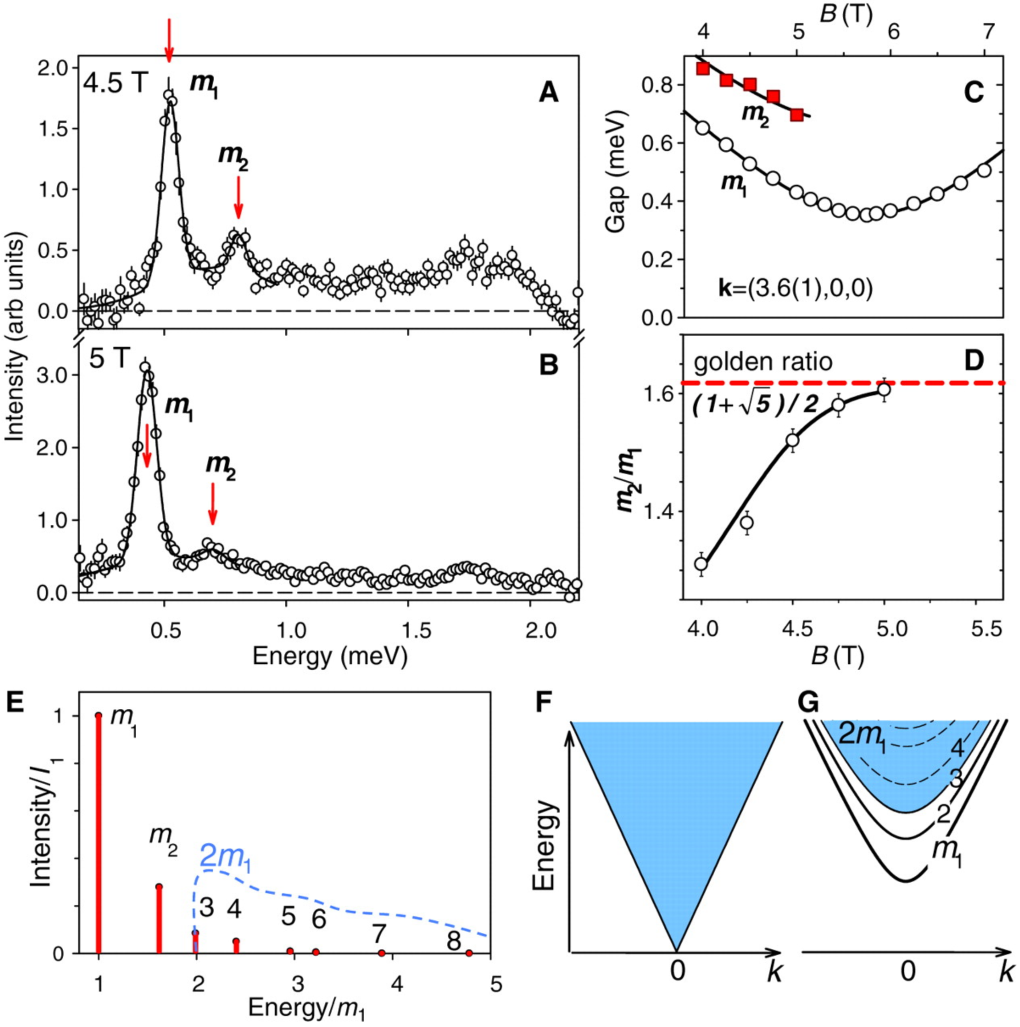
(A and B) Energy scans at the zone center at 4.5 and 5 T observing two peaks, m1 and m2, at low energies.

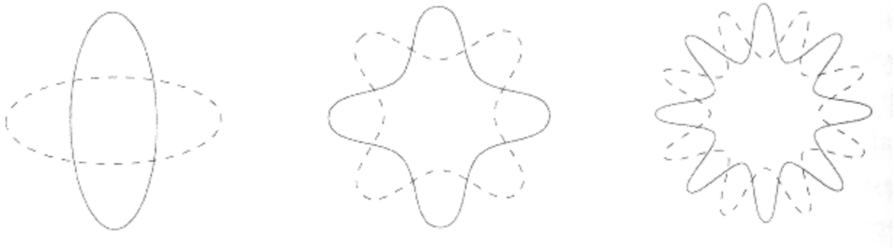


R Coldea et al. Science 2010;327:177-180

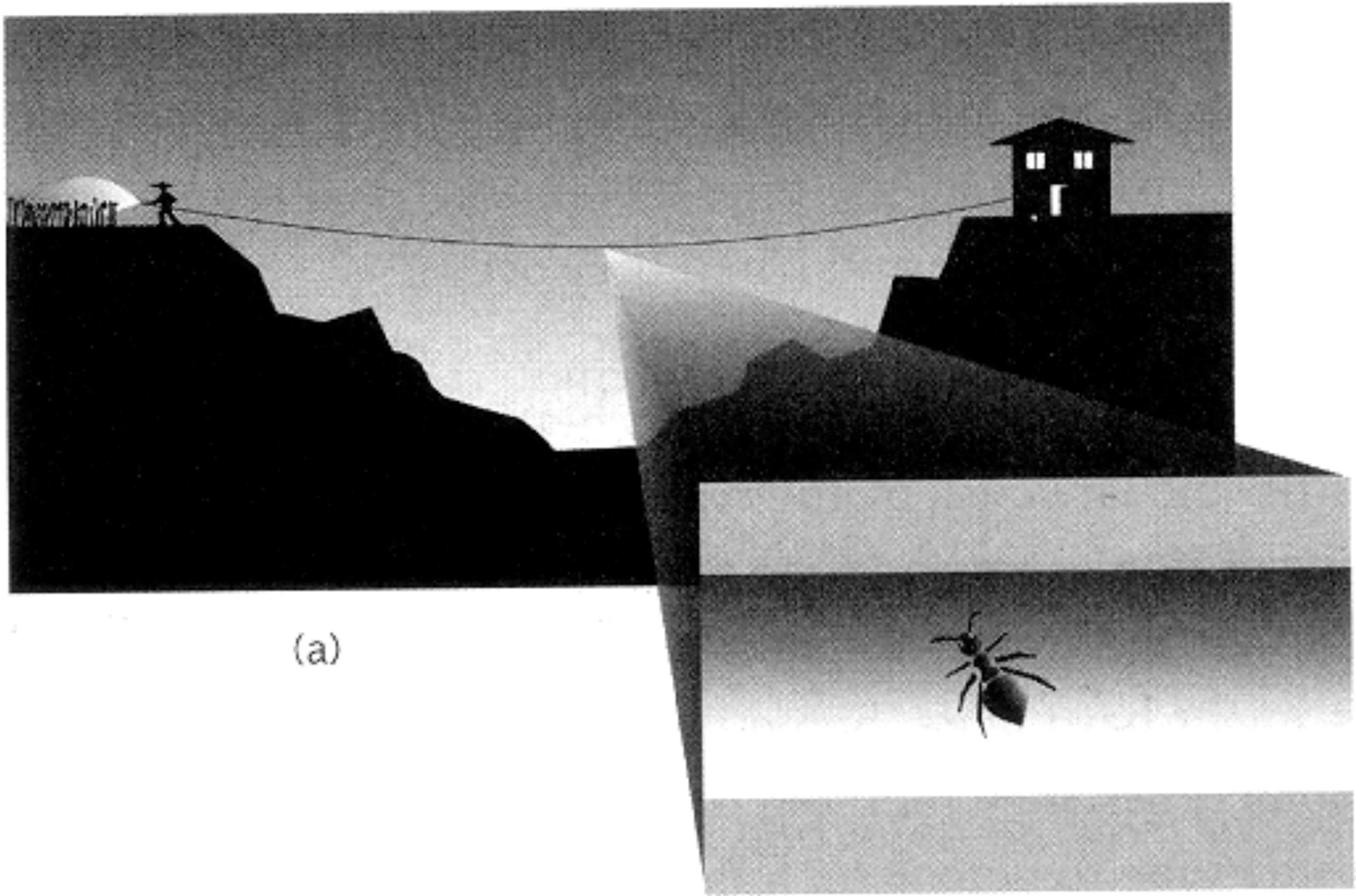
AdS / CFT duality

Quantum gravity

- Planck scale $10^{-35}m$
- Need to quantize gravity
- String theory: a rubber band with Planck size



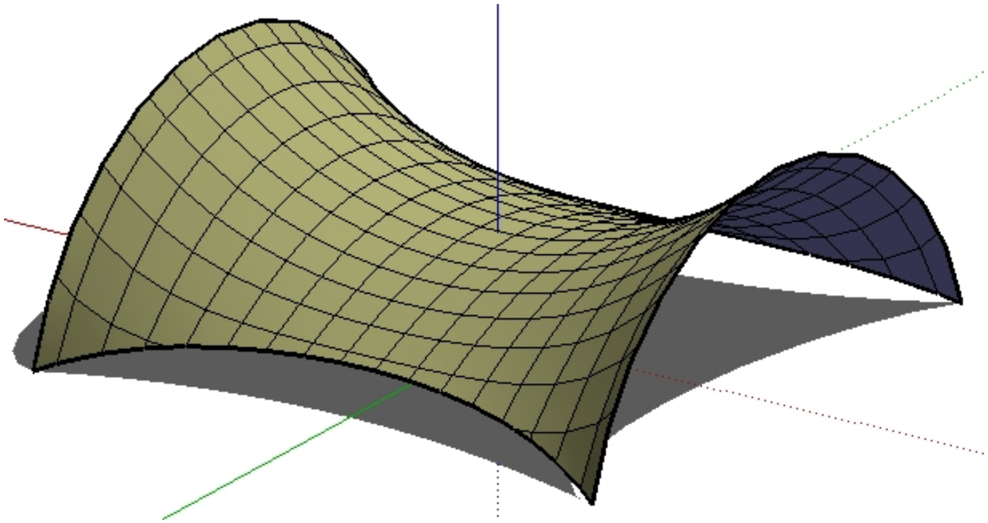
- Only possible in 10-dimensions
  - 6 of them are curled up in very small size



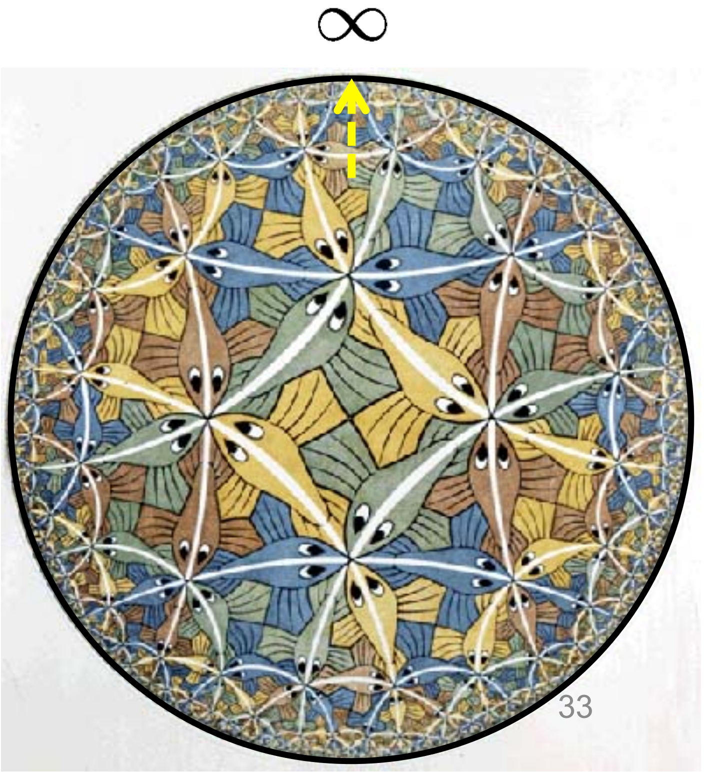
(a)

(b)

Space with negative curvature

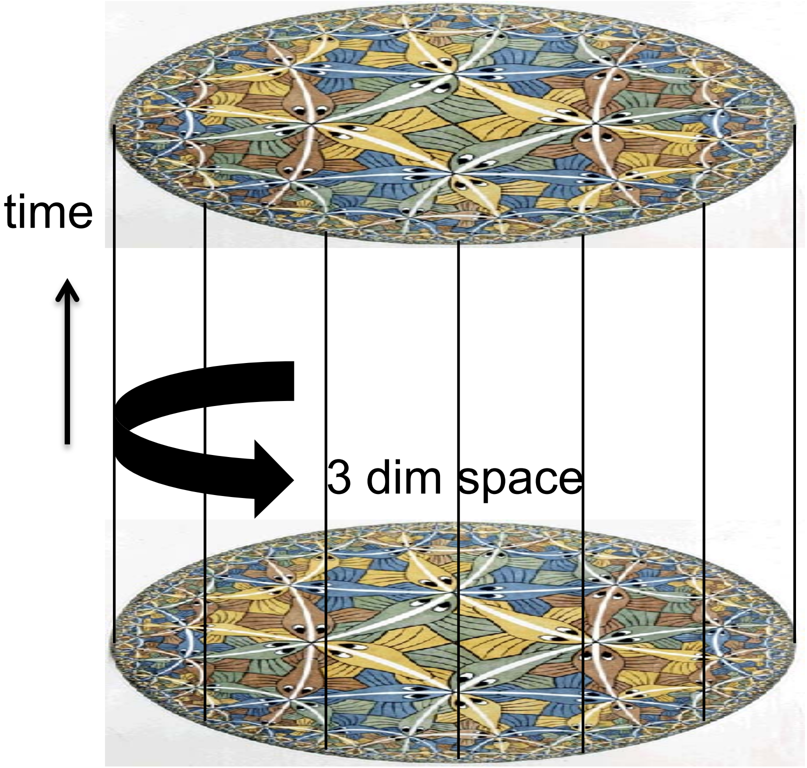


- Anti-de Sitter (AdS) space



AdS space

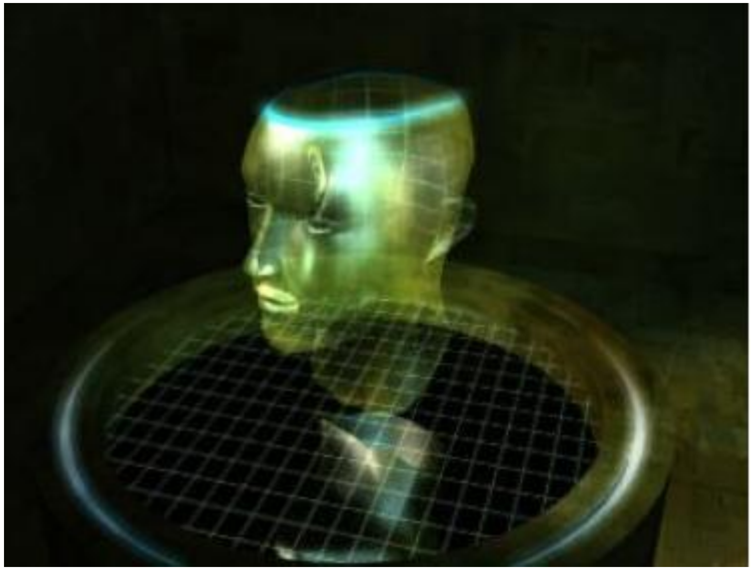
- Inside: 5 dim AdS_5 space



- Boundary: 4 dim space-time

Holography principle

- Hologram

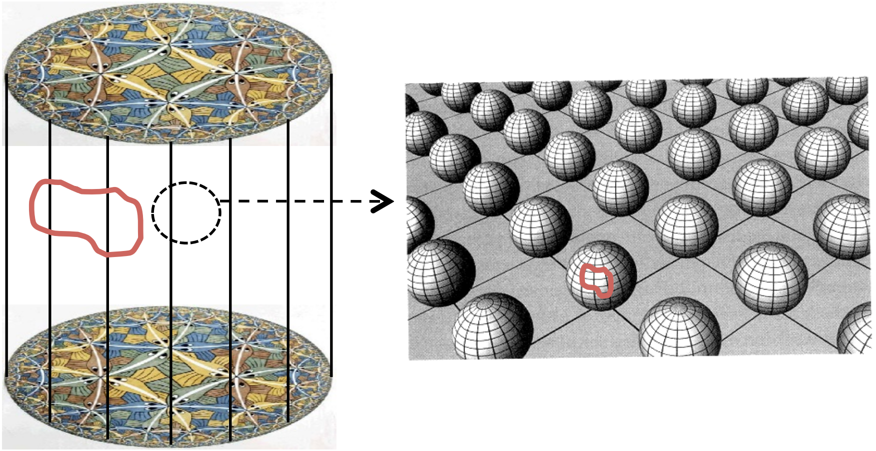


- Holography principle
  - String theory on AdS space = gauge theory on the boundary space-time
  - Quantum gravity effect ←→ gauge interaction

$$\frac{1}{\lambda} \qquad \lambda$$

AdS/CFT conjecture (Maldacena)

- String theory on AdS_5 x S^5



- 4 dim super Yang-Mills theory
  - Scale-invariant theory

$$\beta = 0$$

Gauge theory

- Electromagnetism (Abelian gauge theory)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- Yang-Mills (non-Abelian gauge) theory:

$$A_\mu = N_c \times N_c \text{ Matrix}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] = \frac{1}{2}\sum_a \left(\mathbf{E}_a{}^2 - \mathbf{B}_a{}^2\right)$$

Supersymmetric Yang-Mills theory

- Supersymmetry
  - Scalar boson, fermion, gauge boson


Spin 0 Spin 1/2 Spin 1

  - N=4 SYM

$$(A_\mu, \chi^a_\alpha, \Phi^j), \quad a = 1, \ldots, 4; j = 1, \ldots, 6$$

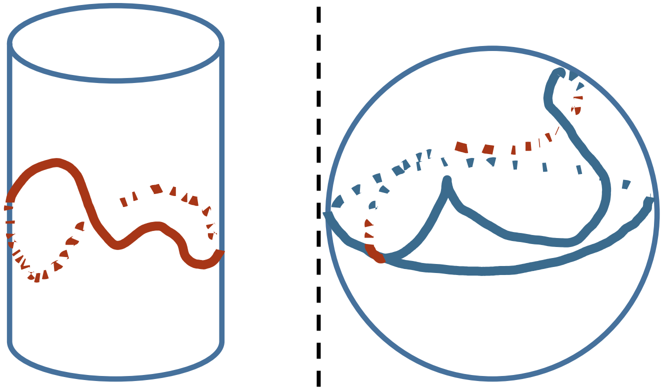
$$S = \frac{\mathrm{Tr}}{g^2_{\mathrm{YM}}} \int d^4x \left\{ -\frac{1}{4} F^2_{\mu\nu} + (D_\mu \Phi^a)^2 + \left[\Phi^a, \Phi^b\right]^2 + \bar{\chi} \not{D} \chi - i \bar{\chi} \Gamma_a [\Phi^a, \chi] \right\}$$

  - RG fixed point: scale invariance

$$\beta \equiv \mu \frac{\partial g_{\mathrm{YM}}}{\partial \mu} = -\frac{g^3_{\mathrm{YM}}}{16\pi^2} \left(\frac{11}{3} N_c - \frac{1}{6} \sum_i^{6N_c} C_i - \frac{1}{3} \sum_j^{8N_c} \tilde{C}_j \right) \qquad = 0$$

AdS/CFT duality

Energy of string = Dimension of SYM



Computable for large λ

$$\mathcal{O}(x) = \mathrm{Tr}\left[XYZF_{\mu\nu}\chi^{\alpha}(D_{\mu}Y)\ldots\right]$$

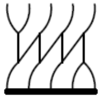
$$\langle\mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{1}{|x|^{2\Delta}}$$

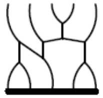
Need non-perturbation

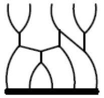
$$\lambda = g_{\mathrm{YM}}^{2}N_{c}$$

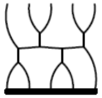
4-Loop su(2) Konishi

$\mathrm{Tr}\,[ZZXX]\,, \quad \mathrm{Tr}\,[ZXZX]$

$F_{A1} =$

$\chi(1,2,3,4)$

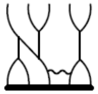
$F_{A2} =$

$\chi(3,2,1,4)$

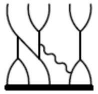
$F_{A3} =$

$\chi(1,4,3,2)$

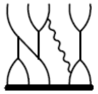
$F_{A4} =$

$\chi(2,4,1,3)$

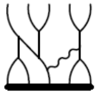
$F_{A5} =$

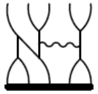
$\chi(2,1,3,2)$

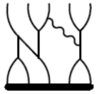
$F_{B1} =$


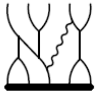
$F_{B2} =$


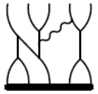
$F_{B3} =$


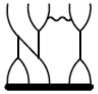
$F_{B4} =$


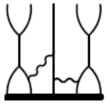
$F_{B5} =$


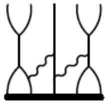
$F_{B6} =$


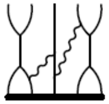
$F_{B7} =$


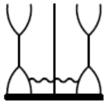
$F_{B8} =$


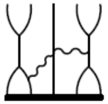
$F_{B9} =$


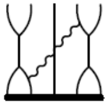
$F_{D1} =$


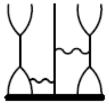
$F_{D2} =$


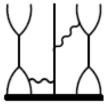
$F_{D3} =$


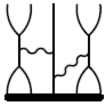
$F_{D4} =$


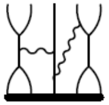
$F_{D5} =$


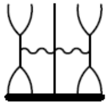
$F_{D6} =$


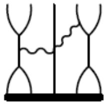
$F_{D7} =$


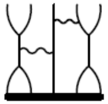
$F_{D8} =$


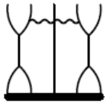
$F_{D9} =$


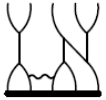
$F_{D10} =$


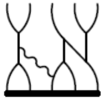
$F_{D11} =$


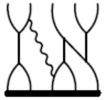
$F_{D12} =$


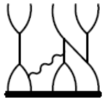
$F_{D13} =$


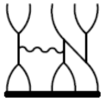
$F_{D14} =$


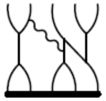
$F_{C1} =$


$F_{C2} =$


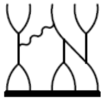
$F_{C3} =$


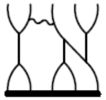
$F_{C4} =$


$F_{C5} =$


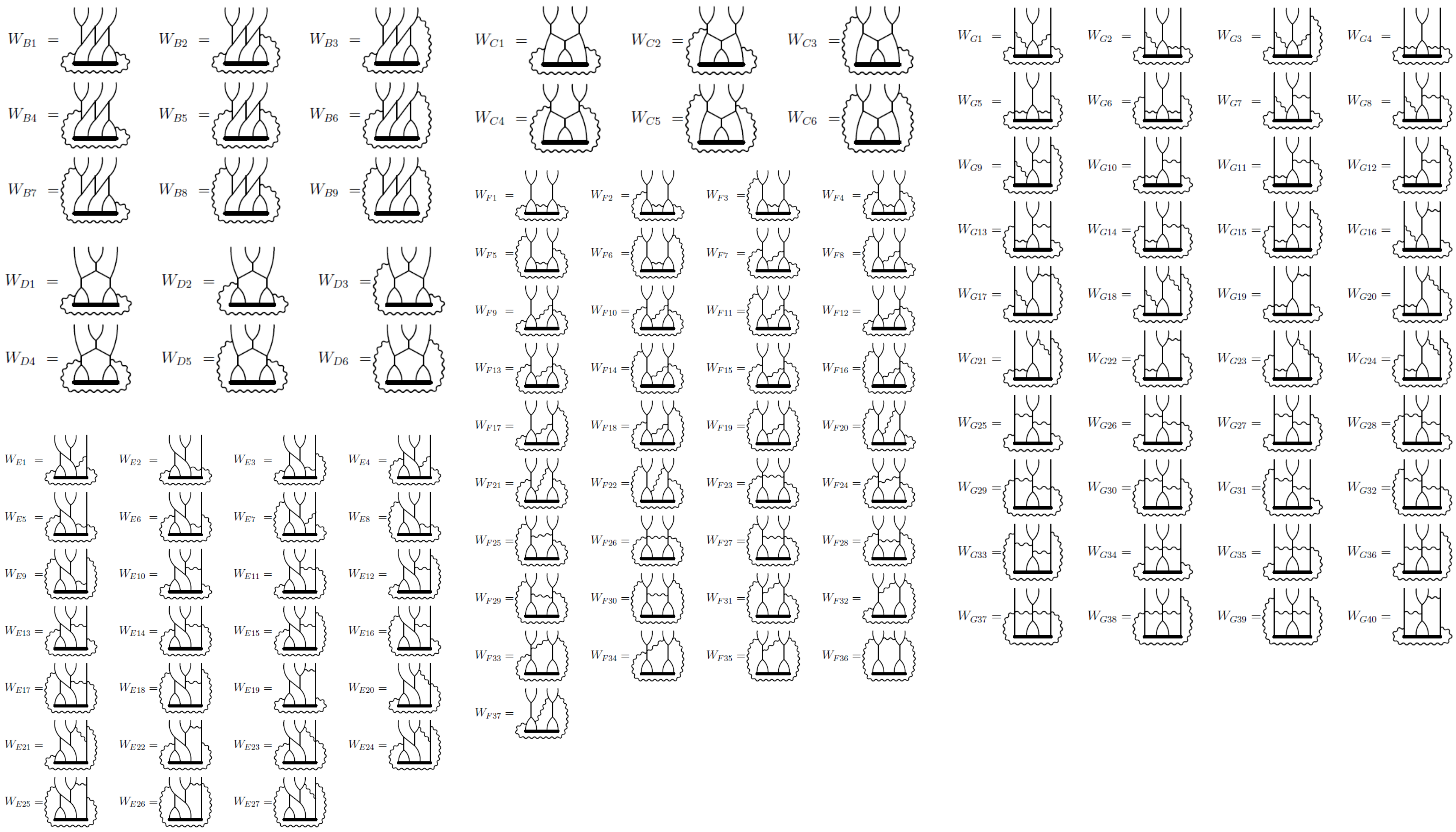
$F_{C6} =$


$F_{C7} =$

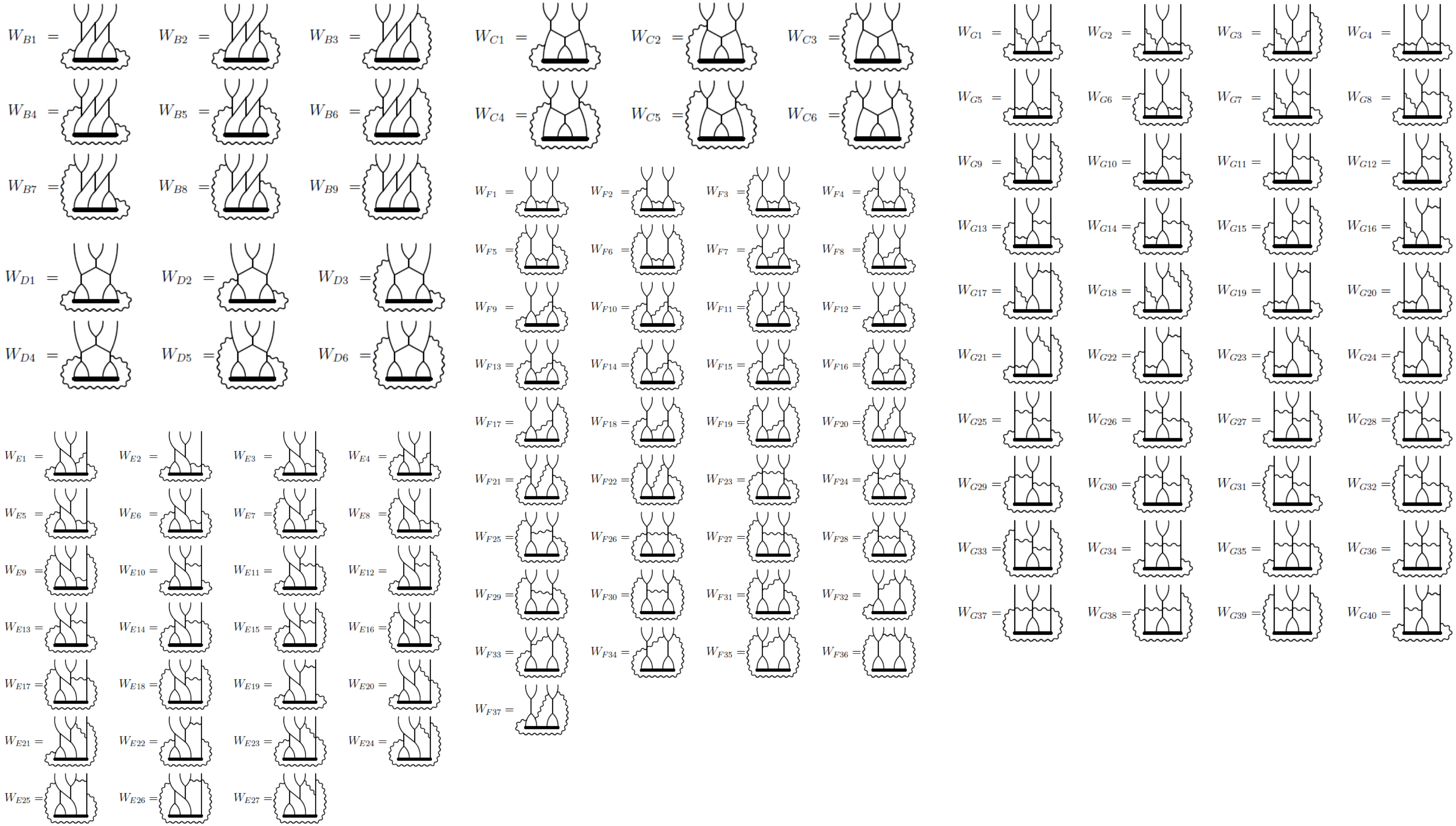

$F_{C8} =$


$F_{C9} =$


Wrapping diagrams



Wrapping diagrams

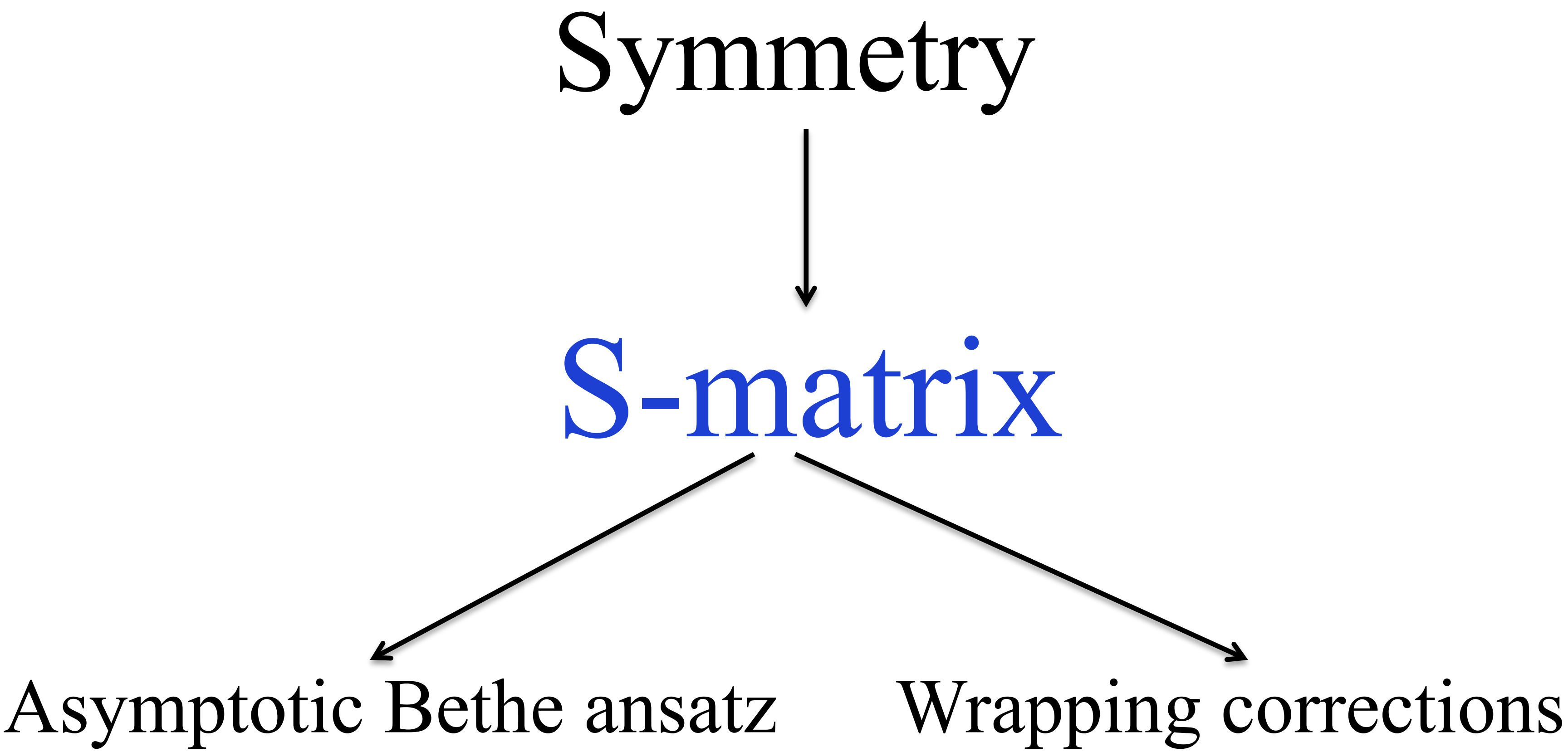


$$\Delta_{\text{Pert.}} = 4+12g^2-48g^4+336g^6-(2496-576\zeta(3)+1440\zeta(5))g^8$$

- Wrapping diagrams

$$\delta\Delta_{\text{wrap}} = \left[324 + 864\zeta(3) - 1440\zeta(5)\right] g^8$$

- Higher order perturbations are not realistic
- Need to solve exactly



Symmetry

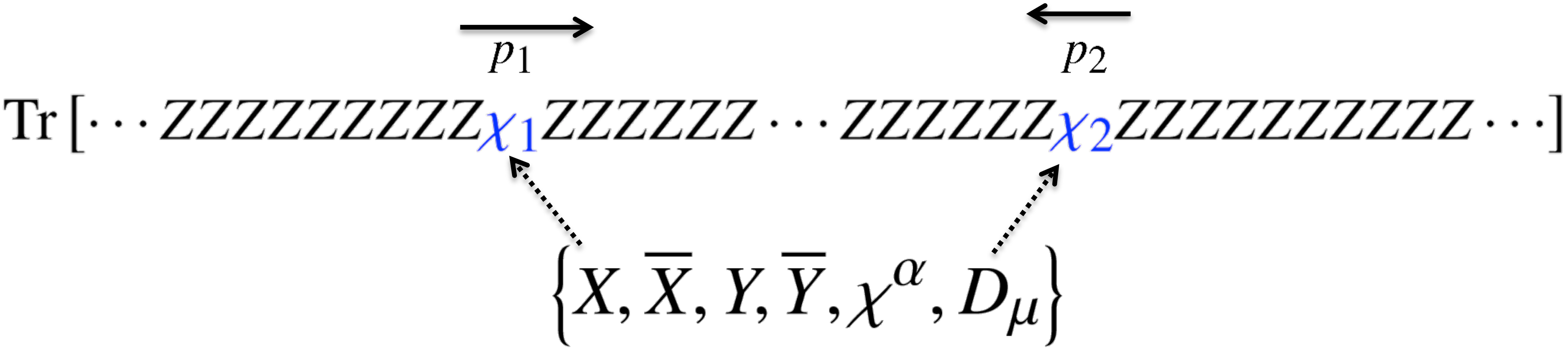
S-matrix

Asymptotic Bethe ansatz

Wrapping corrections

S-matrix

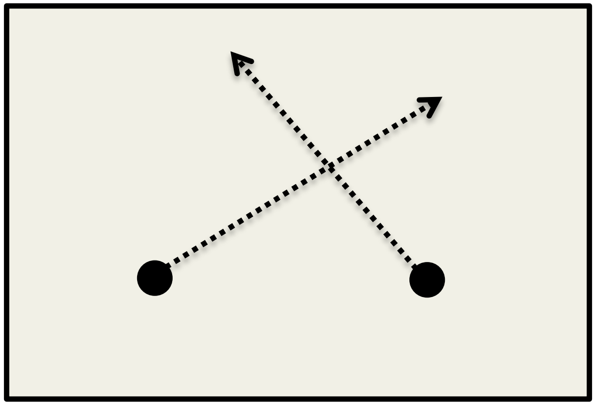
- SYM : scattering of fields on the spin chain



$$\mathrm{Tr}\,[\cdots ZZZZZZZZZZ\chi_1 ZZZZZZ\cdots ZZZZZZ\chi_2 ZZZZZZZZZZZZ\cdots]$$

$$\{X, \overline{X}, Y, \overline{Y}, \chi^\alpha, D_\mu\}$$

- String side : scattering on the world sheet



- Symmetry : 4d conformal group psu(2,2|4)

- From symmetry to S-matrix

$$\left[\mathbf{S}(p_1, p_2), \left(\begin{array}{c|c} \mathbb{L}_a{}^b & \mathbb{Q}_\alpha{}^b \\ \hline \mathbb{Q}_a^{\dagger\beta} & \mathbb{R}_\alpha{}^\beta \end{array}\right)\right] = 0$$

- S : 16 x 16 matrix

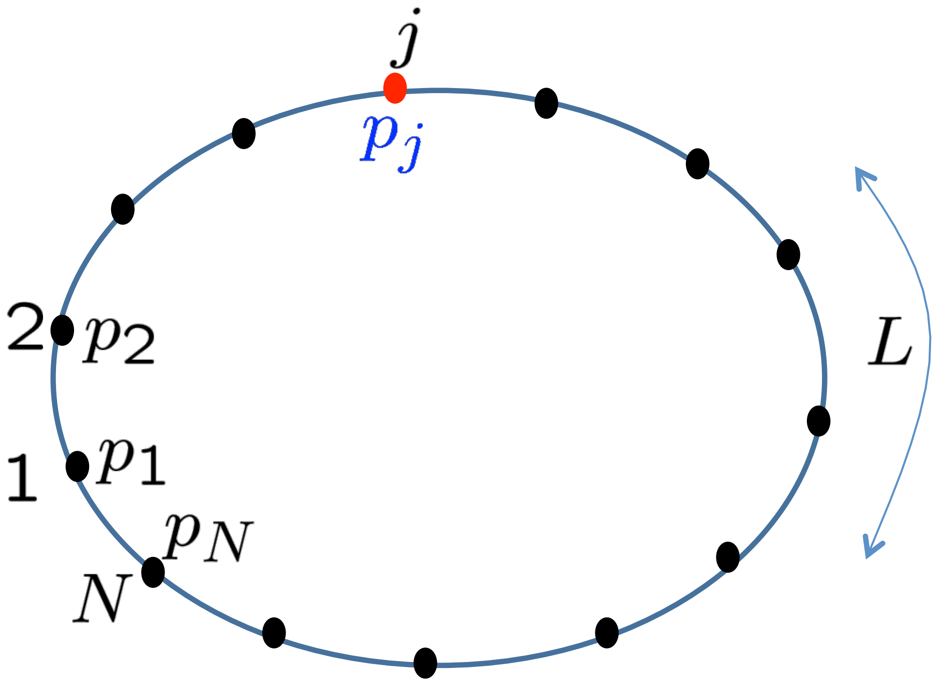
$$\begin{array}{c|c|c} & (ab)(\alpha\beta) & (a\beta)(\alpha b) \\ \hline (ab)(\alpha\beta) & & 0 \\ \hline (a\beta)(\alpha b) & 0 & \end{array}$$

$$\begin{aligned}
S_{a\,a}^{a\,a} &= A\,, \quad S_{\alpha\,\alpha}^{\alpha\,\alpha} = D\,, \\
S_{a\,b}^{a\,b} &= \frac{1}{2}(A-B)\,, \quad S_{a\,b}^{b\,a} = \frac{1}{2}(A+B)\,, \\
S_{\alpha\,\beta}^{\alpha\,\beta} &= \frac{1}{2}(D-E)\,, \quad S_{\alpha\,\beta}^{\beta\,\alpha} = \frac{1}{2}(D+E)\,, \\
S_{a\,b}^{\alpha\,\beta} &= -\frac{1}{2}\epsilon_{ab}\epsilon^{\alpha\beta}C\,, \quad S_{\alpha\,\beta}^{a\,b} = -\frac{1}{2}\epsilon^{ab}\epsilon_{\alpha\beta}F\,, \\
S_{a\,\alpha}^{a\,\alpha} &= G\,, \quad S_{a\,\alpha}^{\alpha\,a} = H\,, \quad S_{\alpha\,a}^{a\,\alpha} = K\,, \quad S_{\alpha\,a}^{\alpha\,a} = L
\end{aligned}$$

$$\begin{aligned}
A &= S_0\frac{x_2^- - x_1^+}{x_2^+ - x_1^-}\frac{\eta_1\eta_2}{\tilde{\eta}_1\tilde{\eta}_2}\,, \\
B &= -S_0\left[\frac{x_2^- - x_1^+}{x_2^+ - x_1^-} + 2\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^-x_2^- - x_1^+x_2^+)}\right]\frac{\eta_1\eta_2}{\tilde{\eta}_1\tilde{\eta}_2}\,, \\
C &= S_0\frac{2ix_1^-x_2^-(x_1^+ - x_2^+)\eta_1\eta_2}{x_1^+x_2^+(x_1^- - x_2^+)(1 - x_1^-x_2^-)}\,, \qquad D = -S_0\,, \\
E &= S_0\left[1 - 2\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- + x_2^+)}{(x_1^- - x_2^+)(x_1^-x_2^- - x_1^+x_2^+)}\right]\,, \\
F &= S_0\frac{2i(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- - x_2^+)(1 - x_1^-x_2^-)\tilde{\eta}_1\tilde{\eta}_2}\,, \\
G &= S_0\frac{(x_2^- - x_1^-)}{(x_2^+ - x_1^-)}\frac{\eta_1}{\tilde{\eta}_1}\,, \qquad H = S_0\frac{(x_2^+ - x_2^-)}{(x_1^- - x_2^+)}\frac{\eta_1}{\tilde{\eta}_2}\,, \\
K &= S_0\frac{(x_1^+ - x_1^-)}{(x_1^- - x_2^+)}\frac{\eta_2}{\tilde{\eta}_1}\,, \qquad L = S_0\frac{(x_1^+ - x_2^+)}{(x_1^- - x_2^+)}\frac{\eta_2}{\tilde{\eta}_2}
\end{aligned}$$

$$\eta_1 = \eta(p_1)e^{ip_2/2}\,, \quad \eta_2 = \eta(p_2)\,, \quad \tilde{\eta}_1 = \eta(p_1)\,, \quad \tilde{\eta}_2 = \eta(p_2)e^{ip_1/2}$$

- Periodic BC



  - At each crossing, S-matrix $$e^{ip_jL}\prod_{k\neq j,1}^{N} S(p_j,p_k)=1$$
  - Formulae for exact dimension can be derived

su(2) Konishi

$$\mathrm{Tr}\,[ZZXX]\,,\qquad \mathrm{Tr}\,[ZXZX]$$

- BAE : $e^{i4p}=e^{-i72\sqrt{3}\zeta(3)g^6}\cdot\frac{2u+i}{2u-i},\qquad u=\frac{1}{2}\cot\frac{p}{2}\sqrt{1+16g^2\sin^2\frac{p}{2}}$

$$p=\frac{2\pi}{3}-\sqrt{3}g^2+\frac{9\sqrt{3}}{2}g^4-\frac{72\sqrt{3}+72\sqrt{3}\zeta(3)}{3}g^6+\dots$$

$$\Delta_{\mathrm{BAE}}=4+12g^2-48g^4+336g^6-(2820+288\zeta(3))g^8+\dots$$

(cf)

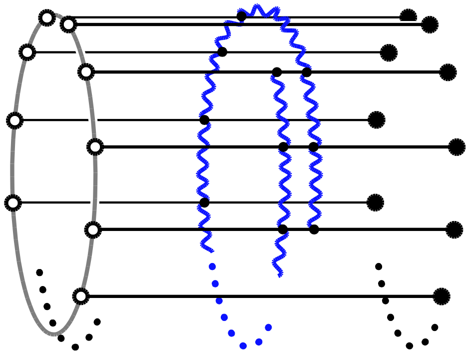
$$\Delta_{\mathrm{Pert.}}=4+12g^2-48g^4+336g^6-(2496-576\zeta(3)+1440\zeta(5))g^8$$

- Discrepancy

$$\delta\Delta=\Delta_{\mathrm{Pert.}}-\Delta_{\mathrm{BAE}}=(324+864\zeta(3)-1440\zeta(5))g^8+\dots$$

Why discrepancy?

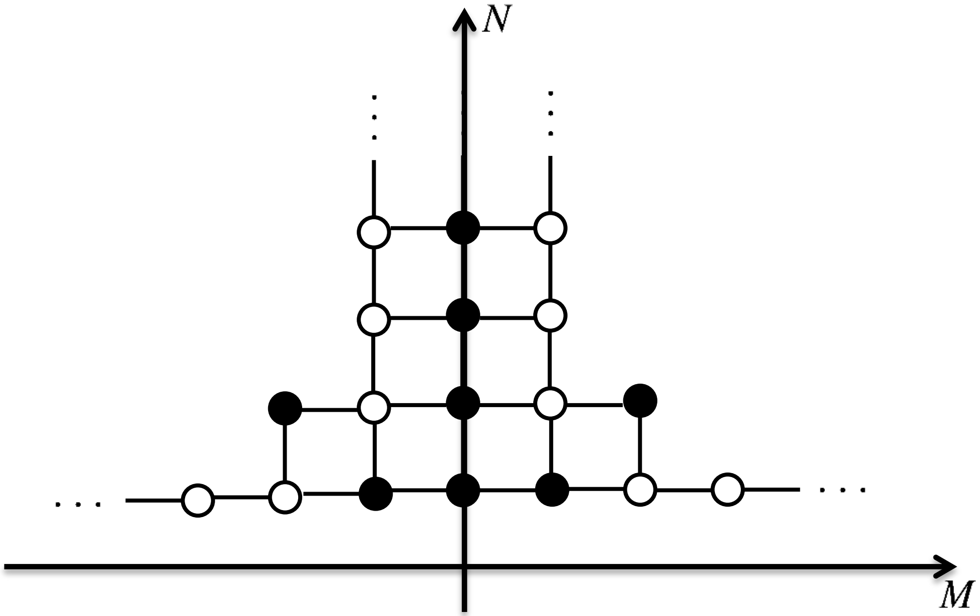
- Because S-matrix is valid only for infinite-size
- Asymptotic Bethe ansatz is valid for infinite length
- Finite-size: Wrapping effects



- Need "finite-size" methods based on S-matrix
  - Luscher correction
  - Thermodynamic Bethe ansatz / Y-system / NLIE

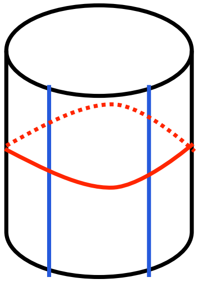
TBA

$$\ln Y_{N,M} = s\star\Big[\ln(1+Y_{N,M+1}) + \ln(1+Y_{N,M-1})\Big] - s\star\Big[\ln(1+Y^{-1}_{N+1,M}) + \ln(1+Y^{-1}_{N-1,M})\Big]$$



$$\Delta = -\sum_{N=1}^{\infty}\int\frac{dp}{2\pi}\log(1+Y_{N,0})$$

Results from TBA / Luscher formula



For the su(2) Konishi [Bajnok, Janik (2008)]

$$\Delta = -\sum_{Q=1}^{\infty}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\underbrace{\frac{4^L g^{2L}}{(Q^2+q^2)^L}\sum_{j,j'}(-1)^{F_{(jj')}}\left[\mathcal{S}^{(Q1)}(q,p)\mathcal{S}^{(Q1)}(q,-p)\right]_{(jj')(11)}^{(jj')(11)}}_{\frac{147456Q^2(3q^3+3Q^2-4)^2}{(q^2+Q^2)^4(9q^4+6[3(Q-2)Q+2]q^2+[3(Q-2)Q+4]^2)}\frac{1}{9q^4+6[3(Q+2)Q+2]q^2+[3(Q+2)Q+4]^2}}$$

L=4

Residue integrals

$$\sum_{Q=1}^{\infty}\left\{-\frac{\mathrm{num}(Q)}{(9Q^4-3Q^2+1)^4(27Q^6-27Q^4+36Q^2+16)}+\frac{864}{Q^3}-\frac{1440}{Q^5}\right\}$$

$$\mathrm{num}(Q)=7776Q(19683Q^{18}-78732Q^{16}+150903Q^{14}-134865Q^{12}+1458Q^{10}+48357Q^8-13311Q^6-1053Q^4+369Q^2-10)$$

Sum $$\delta\Delta = 324 + 864\zeta(3) - 1440\zeta(5)$$

Match exactly!

beta-deformed SYM

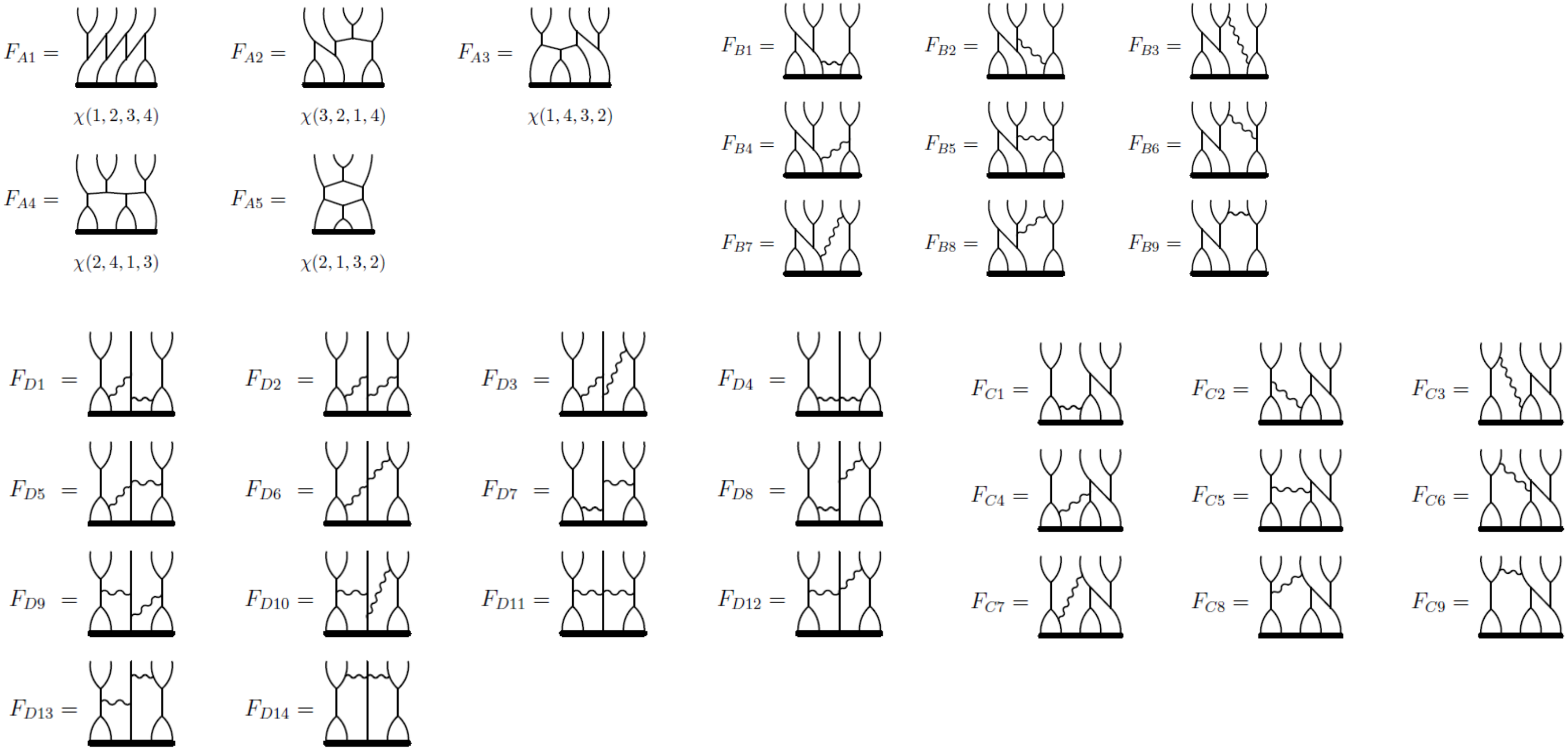
$$V_\beta = \left|e^{i\pi\beta}ZX - e^{-i\pi\beta}XZ\right|^2 + \left|e^{i\pi\beta}XY - e^{-i\pi\beta}YX\right|^2 + \left|e^{i\pi\beta}YZ - e^{-i\pi\beta}ZY\right|^2$$

- $\mathcal{N}$=1 super-CFT
- Dual to string theory on Lunin-Maldacena background $AdS_5 \times S^5_\beta$

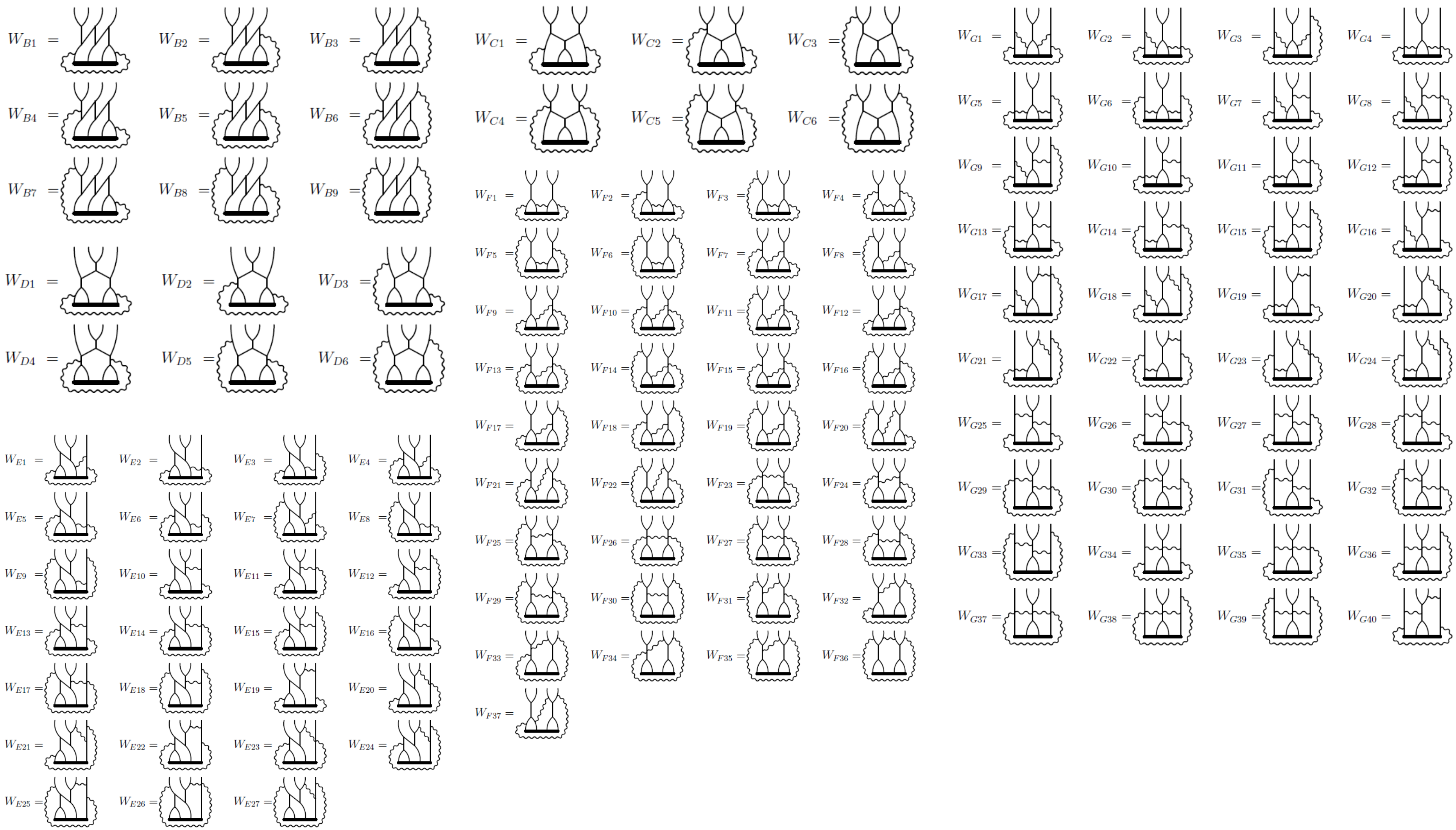
4-Loop su(2) Konishi

[Fiamberti, Santambrogio, Sieg, Zanon (2008)]

N=1 supergraphs



Wrapping diagrams



$$
\begin{aligned}
\gamma &= 4 + g^2\gamma_1 + g^4\gamma_2 + g^6\gamma_3 + g^8\gamma_4 + \dots \\
\gamma_1 &= 6(1+\delta) \\
\gamma_2 &= -\frac{3}{\delta} - 15 - 21\delta - 9\delta^2 \\
\gamma_3 &= -\frac{3}{4\delta^3} + \frac{153}{4\delta} + 114 + \frac{495}{4}\delta + 54\delta^2 + \frac{27}{4}\delta^3 \\
\gamma_4 &= -\frac{3}{8\delta^5} + \frac{33}{2\delta^3} - \frac{1701}{4\delta} - 1230 \\
&- \frac{2427}{2}\delta - 180\delta^2 + 162\delta^4 + \frac{2997}{8}\delta^3 \\
&+ \left(-\frac{9}{\delta} + 297 + 702\delta + 234\delta^2 - 405\delta^3 - 243\delta^4\right)\zeta(3) \\
&- 360\,(1+\delta)^2\,\zeta(5)
\end{aligned}
$$

$$
\delta \equiv \frac{\sqrt{5 + 4\cos(4\pi\beta)}}{3}
$$

S-matrix

[CA-Bajnok-Bombardelli-Nepomechie (2010a)]

- Drinfeld-twist by a constant matrix

$$S_{\beta}(p_1, p_2) = F_{\beta} \cdot S(p_1, p_2) \cdot F_{\beta}$$

- Derived asymptotic BAE from PBC

Beisert-Roiban BAE (2006)

$$
\begin{aligned}
1 &= e^{i\Phi_1(\beta)} \prod_{k=1}^{K_2} \frac{u_{1j}-u_{2k}+\frac{i}{2}}{u_{1j}-u_{2k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{1-g^2/2x_{1j}x_{4k}^+}{1-g^2/2x_{1j}x_{4k}^-} \\
1 &= e^{i\Phi_2(\beta)} \prod_{k=1}^{K_2} \frac{u_{2j}-u_{2k}-i}{u_{2j}-u_{2k}+i} \prod_{k=1}^{K_3} \frac{u_{2j}-u_{3k}+\frac{i}{2}}{u_{2j}-u_{3k}-\frac{i}{2}} \prod_{k=1}^{K_1} \frac{u_{2j}-u_{1k}+\frac{i}{2}}{u_{2j}-u_{1k}-\frac{i}{2}} \\
1 &= e^{i\Phi_3(\beta)} \prod_{k=1}^{K_2} \frac{u_{3j}-u_{2k}+\frac{i}{2}}{u_{3j}-u_{2k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{x_{3j}-x_{4k}^+}{x_{3j}-x_{4k}^-} \\
\left(\frac{x_{4j}^+}{x_{4j}^-}\right)^L &= e^{i\Phi_4(\beta)} \prod_{k=1}^{K_4} \sigma^2(x_{4j},x_{4k}) \frac{x_{4j}^+-x_{4k}^-}{x_{4j}^--x_{4k}^+} \frac{1-g^2/2x_{4j}^+x_{4k}^-}{1-g^2/2x_{4j}^-x_{4k}^+} \\
&\times \prod_{k=1}^{K_1} \frac{1-g^2/2x_{4j}^-x_{1k}}{1-g^2/2x_{4j}^+x_{1k}} \prod_{k=1}^{K_3} \frac{x_{4j}^--x_{3k}}{x_{4j}^+-x_{3k}} \prod_{k=1}^{K_5} \frac{x_{4j}^--x_{5k}}{x_{4j}^+-x_{5k}} \prod_{k=1}^{K_7} \frac{1-g^2/2x_{4j}^-x_{7k}}{1-g^2/2x_{4j}^+x_{7k}} \\
1 &= e^{i\Phi_5(\beta)} \prod_{k=1}^{K_6} \frac{u_{5j}-u_{6k}+\frac{i}{2}}{u_{5j}-u_{6k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{x_{5j}-x_{4k}^+}{x_{5j}-x_{4k}^-} \\
1 &= e^{i\Phi_6(\beta)} \prod_{k=1}^{K_6} \frac{u_{6j}-u_{6k}-i}{u_{6j}-u_{6k}+i} \prod_{k=1}^{K_5} \frac{u_{6j}-u_{5k}+\frac{i}{2}}{u_{6j}-u_{5k}-\frac{i}{2}} \prod_{k=1}^{K_7} \frac{u_{6j}-u_{7k}+\frac{i}{2}}{u_{6j}-u_{7k}-\frac{i}{2}} \\
1 &= e^{i\Phi_7(\beta)} \prod_{k=1}^{K_6} \frac{u_{7j}-u_{6k}+\frac{i}{2}}{u_{7j}-u_{6k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{1-g^2/2x_{7j}x_{4k}^+}{1-g^2/2x_{7j}x_{4k}^-}
\end{aligned}
$$

su(2) Konishi

$$\mathrm{Tr}\,[ZZXX]\,, \quad \mathrm{Tr}\,[ZXZX]$$

- BAE :

$$e^{4ip_1} = e^{8\pi i\beta} \cdot e^{-i72\sqrt{3}\zeta(3)g^6} \cdot \frac{u_1 - u_2 + i}{u_1 - u_2 - i}$$

$$e^{4ip_2} = e^{8\pi i\beta} \cdot e^{i72\sqrt{3}\zeta(3)g^6} \cdot \frac{u_2 - u_1 + i}{u_2 - u_1 - i}$$

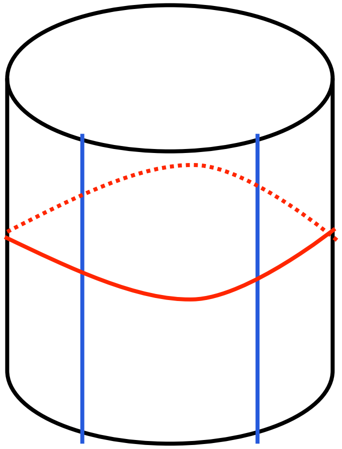
$$\Delta_{\mathrm{BAE}} = 4 + D_1 g^2 + D_2 g^4 + D_3 g^6 + D_4 g^8$$

$$\begin{aligned} D_1 &= 6(1+\delta), \quad D_2 = -\frac{3}{\delta} - 15 - 21\delta - 9\delta^2 \\ D_3 &= -\frac{3}{4\delta^3} + \frac{153}{4\delta} + 114 + \frac{495}{4}\delta + 54\delta^2 + \frac{27}{4}\delta^3 \\ D_4 &= \frac{3\,(1+\delta)^4}{8\delta^5(1+3\delta)^2}(-1 - 2\delta + 49\delta^2 + 84\delta^3 \\ &- 1359\delta^4 - 5562\delta^5 - 2673\delta^6 + 1944\delta^7) \\ &+ \left(-\frac{9}{\delta} + 27 + 54\delta - 90\delta^2 - 189\delta^3 - 81\delta^4\right)\zeta(3) \end{aligned}$$

- Discrepancy

$$\delta\Delta = \Delta_{\text{Pert.}} - \Delta_{\text{BAE}}$$
$$= g^8\left[-54(1+\delta)^3(-5+3\delta)\zeta(3) - 360(1+\delta)^2\zeta(5) + \frac{81\,(1-3\,\delta)^2\,(1+\delta)^4}{(1+3\,\delta)^2}\right]$$

- Need to be explained by TBA/Luscher based on the S-matrix



Results from TBA / Luscher formula

[CA-Bajnok-Bombardelli-Nepomechie (2010b)]

$$\Delta = -\sum_{Q=1}^{\infty}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\,\frac{4^L g^{2L}}{(Q^2+q^2)^L}\,\sum_{j,j'}(-1)^{F_{(jj')}}\left[\mathcal{S}^{(Q1)}(q,p)\mathcal{S}^{(Q1)}(q,-p)\right]_{(jj')(11)}^{(jj')(11)}$$

After residue integrals

$$\sum_{Q=1}^{\infty}\left[\frac{f_1}{Q^5}+\frac{f_2}{Q^3}+f_3(Q)\right]$$

$$\begin{aligned}
f_1 &= -\frac{2560(1+2u_1^2+2u_2^2)^2}{(4u_1^2+1)^2(4u_2^2+1)^2}\\
f_2 &= \frac{\mathsf{num}}{(4u_1^2+1)^4(4u_2^2+1)^4}\\
\mathsf{num} &= 2048\Big(-1+5u_1^2+48u_1^4+96u_1^6-2u_1u_2-16u_1^3u_2-32u_1^5u_2+5u_2^2+224u_1^2u_2^2\\
&+ 1024u_1^4u_2^2+1536u_1^6u_2^2+768u_1^8u_2^2-16u_1u_2^3-128u_1^3u_2^3+64u_1^8\\
&- 256u_1^5u_2^3+48u_2^4+1024u_1^2u_2^4+3200u_1^4u_2^4+2560u_1^6u_2^4-32u_1u_2^5\\
&- 256u_1^3u_2^5-512u_1^5u_2^5+96u_2^6+1536u_1^2u_2^6+2560u_1^4u_2^6+64u_2^8+768u_1^2u_2^8\Big)
\end{aligned}$$

$$f_3 = \mathsf{1775\ terms}$$

$$u_{1,2} = \frac{(1-3\,\delta)^2}{2\sqrt{-1+9\,\delta^2}\left(3\sqrt{1-\delta^2}\pm 2\sqrt{\frac{3+3\delta}{1+3\,\delta}}\right)}$$

$$
\begin{aligned}
&-(512\, i\, (-5\, i + 156\, i\, Q^{2} - 2120\, i\, Q^{4} + 16512\, i\, Q^{6} - 77952\, i\, Q^{8} + 216064\, i\, Q^{10} - 317440\, i\, Q^{12} + 196608\, i\, Q^{14} - 32768\, i\, Q^{16} - 160\, Q\, u1 + 4352\, Q^{3}\, u1 - \\
&\quad 50432\, Q^{5}\, u1 + 321408\, Q^{7}\, u1 - 1178624\, Q^{9}\, u1 + 2396160\, Q^{11}\, u1 - 2490368\, Q^{13}\, u1 + 1277952\, Q^{15}\, u1 - 262144\, Q^{17}\, u1 - 140\, i\, u1^{2} + \\
&\quad 6036\, i\, Q^{2}\, u1^{2} - 97280\, i\, Q^{4}\, u1^{2} + 813504\, i\, Q^{6}\, u1^{2} - 3904512\, i\, Q^{8}\, u1^{2} + 10729472\, i\, Q^{10}\, u1^{2} - 16007168\, i\, Q^{12}\, u1^{2} + 12238848\, i\, Q^{14}\, u1^{2} - \\
&\quad 4456448\, i\, Q^{16}\, u1^{2} + 786432\, i\, Q^{18}\, u1^{2} - 3840\, Q\, u1^{3} + 109184\, Q^{3}\, u1^{3} - 1286656\, Q^{5}\, u1^{3} + 8186112\, Q^{7}\, u1^{3} - 29806080\, Q^{9}\, u1^{3} + 60510208\, Q^{11}\, u1^{3} - \\
&\quad 65617920\, Q^{13}\, u1^{3} + 38076416\, Q^{15}\, u1^{3} - 11403264\, Q^{17}\, u1^{3} + 1048576\, Q^{19}\, u1^{3} - 1700\, i\, u1^{4} + 85248\, i\, Q^{2}\, u1^{4} - 1436928\, i\, Q^{4}\, u1^{4} + \\
&\quad 12094720\, i\, Q^{6}\, u1^{4} - 57501696\, i\, Q^{8}\, u1^{4} + 155193344\, i\, Q^{10}\, u1^{4} - 229015552\, i\, Q^{12}\, u1^{4} + 182779904\, i\, Q^{14}\, u1^{4} - 75235328\, i\, Q^{16}\, u1^{4} + \\
&\quad 13631488\, i\, Q^{18}\, u1^{4} - 39040\, Q\, u1^{5} + 1111040\, Q^{3}\, u1^{5} - 12920832\, Q^{5}\, u1^{5} + 80328704\, Q^{7}\, u1^{5} - 284033024\, Q^{9}\, u1^{5} + 555851776\, Q^{11}\, u1^{5} - \\
&\quad 584318976\, Q^{13}\, u1^{5} + 322699264\, Q^{15}\, u1^{5} - 83886080\, Q^{17}\, u1^{5} + 2097152\, Q^{19}\, u1^{5} - 11680\, i\, u1^{6} + 614912\, i\, Q^{2}\, u1^{6} - 10256384\, i\, Q^{4}\, u1^{6} + \\
&\quad 83934720\, i\, Q^{6}\, u1^{6} - 384442368\, i\, Q^{8}\, u1^{6} + 984932352\, i\, Q^{10}\, u1^{6} - 1352138752\, i\, Q^{12}\, u1^{6} + 976879616\, i\, Q^{14}\, u1^{6} - 331350016\, i\, Q^{16}\, u1^{6} + \\
&\quad 27262976\, i\, Q^{18}\, u1^{6} - 217600\, Q\, u1^{7} + 5965824\, Q^{3}\, u1^{7} - 66535424\, Q^{5}\, u1^{7} + 393981952\, Q^{7}\, u1^{7} - 1310375936\, Q^{9}\, u1^{7} + 2346057728\, Q^{11}\, u1^{7} - \\
&\quad 2179596288\, Q^{13}\, u1^{7} + 960495616\, Q^{15}\, u1^{7} - 148897792\, Q^{17}\, u1^{7} - 49600\, i\, u1^{8} + 2532608\, i\, Q^{2}\, u1^{8} - 40226816\, i\, Q^{4}\, u1^{8} + 310726656\, i\, Q^{6}\, u1^{8} - \\
&\quad 1330298880\, i\, Q^{8}\, u1^{8} + 3100934144\, i\, Q^{10}\, u1^{8} - 3700162560\, i\, Q^{12}\, u1^{8} + 2134900736\, i\, Q^{14}\, u1^{8} - 486539264\, i\, Q^{16}\, u1^{8} + 8388608\, i\, Q^{18}\, u1^{8} - \\
&\quad 716800\, Q\, u1^{9} + 18268160\, Q^{3}\, u1^{9} - 189792256\, Q^{5}\, u1^{9} + 1041408000\, Q^{7}\, u1^{9} - 3144482816\, Q^{9}\, u1^{9} + 4859232256\, Q^{11}\, u1^{9} - 3649044480\, Q^{13}\, u1^{9} + \\
&\quad 1115684864\, Q^{15}\, u1^{9} - 67108864\, Q^{17}\, u1^{9} - 133120\, i\, u1^{10} + 6116352\, i\, Q^{2}\, u1^{10} - 89391104\, i\, Q^{4}\, u1^{10} + 633085952\, i\, Q^{6}\, u1^{10} - \\
&\quad 2452226048\, i\, Q^{8}\, u1^{10} + 4944297984\, i\, Q^{10}\, u1^{10} - 4777312256\, i\, Q^{12}\, u1^{10} + 1935671296\, i\, Q^{14}\, u1^{10} - 234881024\, i\, Q^{16}\, u1^{10} - 1392640\, Q\, u1^{11} + \\
&\quad 31883264\, Q^{3}\, u1^{11} - 298713088\, Q^{5}\, u1^{11} + 1471610880\, Q^{7}\, u1^{11} - 3861774336\, Q^{9}\, u1^{11} + 4783603712\, Q^{11}\, u1^{11} - 2573205504\, Q^{13}\, u1^{11} + \\
&\quad 469762048\, Q^{15}\, u1^{11} - 220160\, i\, u1^{12} + 8364032\, i\, Q^{2}\, u1^{12} - 108068864\, i\, Q^{4}\, u1^{12} + 678658048\, i\, Q^{6}\, u1^{12} - 2279079936\, i\, Q^{8}\, u1^{12} + \\
&\quad 3649568768\, i\, Q^{10}\, u1^{12} - 2550136832\, i\, Q^{12}\, u1^{12} + 587202560\, i\, Q^{14}\, u1^{12} - 1474560\, Q\, u1^{13} + 29360128\, Q^{3}\, u1^{13} - 237502464\, Q^{5}\, u1^{13} + \\
&\quad 998375424\, Q^{7}\, u1^{13} - 2085617664\, Q^{9}\, u1^{13} + 1797259264\, Q^{11}\, u1^{13} - 469762048\, Q^{13}\, u1^{13} - 204800\, i\, u1^{14} + 5734400\, i\, Q^{2}\, u1^{14} - \\
&\quad 60817408\, i\, Q^{4}\, u1^{14} + 319029248\, i\, Q^{6}\, u1^{14} - 851443712\, i\, Q^{8}\, u1^{14} + 843055104\, i\, Q^{10}\, u1^{14} - 234881024\, i\, Q^{12}\, u1^{14} - 655360\, Q\, u1^{15} + \\
&\quad 11010048\, Q^{3}\, u1^{15} - 71303168\, Q^{5}\, u1^{15} + 222298112\, Q^{7}\, u1^{15} - 234881024\, Q^{9}\, u1^{15} + 67108864\, Q^{11}\, u1^{15} - 81920\, i\, u1^{16} + 1376256\, i\, Q^{2}\, u1^{16} - \\
&\quad 8912896\, i\, Q^{4}\, u1^{16} + 27787264\, i\, Q^{6}\, u1^{16} - 29360128\, i\, Q^{8}\, u1^{16} + 8388608\, i\, Q^{10}\, u1^{16} - 160\, Q\, u2 + 4352\, Q^{3}\, u2 - 50432\, Q^{5}\, u2 + 321408\, Q^{7}\, u2 - \\
&\quad 1178624\, Q^{9}\, u2 + 2396160\, Q^{11}\, u2 - 2490368\, Q^{13}\, u2 + 1277952\, Q^{15}\, u2 - 262144\, Q^{17}\, u2 + 5112\, i\, Q^{2}\, u1\, u2 - 118656\, i\, Q^{4}\, u1\, u2 + 1138048\, i\, Q^{6}\, u1\, u2 + \\
&\quad 5663744\, i\, Q^{8}\, u1\, u2 + 15286272\, i\, Q^{10}\, u1\, u2 - 22249472\, i\, Q^{12}\, u1\, u2 + 17924096\, i\, Q^{14}\, u1\, u2 - 7602176\, i\, Q^{16}\, u1\, u2 + 1572864\, i\, Q^{18}\, u1\, u2 - \\
&\quad 4480\, Q\, u1^{2}\, u2 + 174976\, Q^{3}\, u1^{2}\, u2 - 2408960\, Q^{5}\, u1^{2}\, u2 + 16281344\, Q^{7}\, u1^{2}\, u2 - 59152896\, Q^{9}\, u1^{2}\, u2 + 117698560\, Q^{11}\, u1^{2}\, u2 - \\
&\quad 129744896\, Q^{13}\, u1^{2}\, u2 + 79364096\, Q^{15}\, u1^{2}\, u2 - 26869760\, Q^{17}\, u1^{2}\, u2 + 3145728\, Q^{19}\, u1^{2}\, u2 + 122688\, i\, Q^{2}\, u1^{3}\, u2 - 2995968\, i\, Q^{4}\, u1^{3}\, u2 + \\
&\quad 29113856\, i\, Q^{6}\, u1^{3}\, u2 - 143710208\, i\, Q^{8}\, u1^{3}\, u2 + 382468096\, i\, Q^{10}\, u1^{3}\, u2 - 560791552\, i\, Q^{12}\, u1^{3}\, u2 + 468844544\, i\, Q^{14}\, u1^{3}\, u2 - \\
&\quad 220725248\, i\, Q^{16}\, u1^{3}\, u2 + 46137344\, i\, Q^{18}\, u1^{3}\, u2 - 54400\, Q\, u1^{4}\, u2 + 2505216\, Q^{3}\, u1^{4}\, u2 - 35850240\, Q^{5}\, u1^{4}\, u2 + 241755136\, Q^{7}\, u1^{4}\, u2 - \\
&\quad 858062848\, Q^{9}\, u1^{4}\, u2 + 1658478592\, Q^{11}\, u1^{4}\, u2 - 1806041088\, Q^{13}\, u1^{4}\, u2 + 1094189056\, Q^{15}\, u1^{4}\, u2 - 325058560\, Q^{17}\, u1^{4}\, u2 + \\
&\quad 14680064\, Q^{19}\, u1^{4}\, u2 + 1247360\, i\, Q^{2}\, u1^{5}\, u2 - 30472192\, i\, Q^{4}\, u1^{5}\, u2 + 290317312\, i\, Q^{6}\, u1^{5}\, u2 - 1384857600\, i\, Q^{8}\, u1^{5}\, u2 + \\
&\quad 3517972480\, i\, Q^{10}\, u1^{5}\, u2 - 4901568512\, i\, Q^{12}\, u1^{5}\, u2 + 3774087168\, i\, Q^{14}\, u1^{5}\, u2 - 1488977920\, i\, Q^{16}\, u1^{5}\, u2 + 167772160\, i\, Q^{18}\, u1^{5}\, u2 - \\
&\quad 373760\, Q\, u1^{6}\, u2 + 18141184\, Q^{3}\, u1^{6}\, u2 - 254623744\, Q^{5}\, u1^{6}\, u2 + 1650380800\, Q^{7}\, u1^{6}\, u2 - 5530009600\, Q^{9}\, u1^{6}\, u2 + 9881518080\, Q^{11}\, u1^{6}\, u2 - \\
&\quad 9639165952\, Q^{13}\, u1^{6}\, u2 + 4859101184\, Q^{15}\, u1^{6}\, u2 - 945815552\, Q^{17}\, u1^{6}\, u2 + 33554432\, Q^{19}\, u1^{6}\, u2 + 6952960\, i\, Q^{2}\, u1^{7}\, u2 - 162570240\, i\, Q^{4}\, u1^{7}\, u2 + \\
&\quad 1470644224\, i\, Q^{6}\, u1^{7}\, u2 - 6562283520\, i\, Q^{8}\, u1^{7}\, u2 + 15195176960\, i\, Q^{10}\, u1^{7}\, u2 - 18733858816\, i\, Q^{12}\, u1^{7}\, u2 + 11924406272\, i\, Q^{14}\, u1^{7}\, u2 - \\
&\quad 3439329280\, i\, Q^{16}\, u1^{7}\, u2 + 301989888\, i\, Q^{18}\, u1^{7}\, u2 - 1587200\, Q\, u1^{8}\, u2 + 74530816\, Q^{3}\, u1^{8}\, u2 - 984940544\, Q^{5}\, u1^{8}\, u2 + \\
&\quad 5928378368\, Q^{7}\, u1^{8}\, u2 - 18003984384\, Q^{9}\, u1^{8}\, u2 + 28163702784\, Q^{11}\, u1^{8}\, u2 - 22705864704\, Q^{13}\, u1^{8}\, u2 + 8858370048\, Q^{15}\, u1^{8}\, u2 - \\
&\quad 1325400064\, Q^{17}\, u1^{8}\, u2 + 33554432\, Q^{19}\, u1^{8}\, u2 + 22906880\, i\, Q^{2}\, u1^{9}\, u2 - 491487232\, i\, Q^{4}\, u1^{9}\, u2 + 4084072448\, i\, Q^{6}\, u1^{9}\, u2 - \\
&\quad 16451633152\, i\, Q^{8}\, u1^{9}\, u2 + 33054523392\, i\, Q^{10}\, u1^{9}\, u2 - 33852227584\, i\, Q^{12}\, u1^{9}\, u2 + 16949182464\, i\, Q^{14}\, u1^{9}\, u2 - 3758096384\, i\, Q^{16}\, u1^{9}\, u2 + \\
&\quad 268435456\, i\, Q^{18}\, u1^{9}\, u2 - 4259840\, Q\, u1^{10}\, u2 + 178356224\, Q^{3}\, u1^{10}\, u2 - 2139226112\, Q^{5}\, u1^{10}\, u2 + 11541086208\, Q^{7}\, u1^{10}\, u2 -
\end{aligned}
$$

$30298210304\, Q^{9}\, u1^{10}\, u2 + 39352008704\, Q^{11}\, u1^{10}\, u2 - 24798822400\, Q^{13}\, u1^{10}\, u2 + 7532969984\, Q^{15}\, u1^{10}\, u2 - 939524096\, Q^{17}\, u1^{10}\, u2 +$
$44515328\, i\, Q^{2}\, u1^{11}\, u2 - 840368128\, i\, Q^{4}\, u1^{11}\, u2 + 6171262976\, i\, Q^{6}\, u1^{11}\, u2 - 21483225088\, i\, Q^{8}\, u1^{11}\, u2 + 35381051392\, i\, Q^{10}\, u1^{11}\, u2 -$
$28059893760\, i\, Q^{12}\, u1^{11}\, u2 + 10905190400\, i\, Q^{14}\, u1^{11}\, u2 - 1879048192\, i\, Q^{16}\, u1^{11}\, u2 - 7045120\, Q\, u1^{12}\, u2 + 239206400\, Q^{3}\, u1^{12}\, u2 -$
$2496135168\, Q^{5}\, u1^{12}\, u2 + 11556880384\, Q^{7}\, u1^{12}\, u2 - 24384634880\, Q^{9}\, u1^{12}\, u2 + 24289214464\, Q^{11}\, u1^{12}\, u2 - 11291066368\, Q^{13}\, u1^{12}\, u2 +$
$2348810240\, Q^{15}\, u1^{12}\, u2 + 47153152\, i\, Q^{2}\, u1^{13}\, u2 - 750256128\, i\, Q^{4}\, u1^{13}\, u2 + 4593025024\, i\, Q^{6}\, u1^{13}\, u2 - 12696158208\, i\, Q^{8}\, u1^{13}\, u2 +$
$15523119104\, i\, Q^{10}\, u1^{13}\, u2 - 8120172544\, i\, Q^{12}\, u1^{13}\, u2 + 1879048192\, i\, Q^{14}\, u1^{13}\, u2 - 6553600\, Q\, u1^{14}\, u2 + 157286400\, Q^{3}\, u1^{14}\, u2 -$
$1317011456\, Q^{5}\, u1^{14}\, u2 + 4816109568\, Q^{7}\, u1^{14}\, u2 - 6861881344\, Q^{9}\, u1^{14}\, u2 + 3841982464\, Q^{11}\, u1^{14}\, u2 - 939524096\, Q^{13}\, u1^{14}\, u2 +$
$20971520\, i\, Q^{2}\, u1^{15}\, u2 - 268435456\, i\, Q^{4}\, u1^{15}\, u2 + 1207959552\, i\, Q^{6}\, u1^{15}\, u2 - 1862270976\, i\, Q^{8}\, u1^{15}\, u2 + 1073741824\, i\, Q^{10}\, u1^{15}\, u2 -$
$268435456\, i\, Q^{12}\, u1^{15}\, u2 - 2621440\, Q\, u1^{16}\, u2 + 33554432\, Q^{3}\, u1^{16}\, u2 - 150994944\, Q^{5}\, u1^{16}\, u2 + 232783872\, Q^{7}\, u1^{16}\, u2 - 134217728\, Q^{9}\, u1^{16}\, u2 +$
$33554432\, Q^{11}\, u1^{16}\, u2 - 140\, i\, u2^{2} + 6036\, i\, Q^{2}\, u2^{2} - 97280\, i\, Q^{4}\, u2^{2} + 813504\, i\, Q^{6}\, u2^{2} - 3904512\, i\, Q^{8}\, u2^{2} + 10729472\, i\, Q^{10}\, u2^{2} -$
$16007168\, i\, Q^{12}\, u2^{2} + 12238848\, i\, Q^{14}\, u2^{2} - 4456448\, i\, Q^{16}\, u2^{2} + 786432\, i\, Q^{18}\, u2^{2} - 4480\, Q\, u1\, u2^{2} + 174976\, Q^{3}\, u1\, u2^{2} - 2408960\, Q^{5}\, u1\, u2^{2} +$
$16281344\, Q^{7}\, u1\, u2^{2} - 59152896\, Q^{9}\, u1\, u2^{2} + 117698560\, Q^{11}\, u1\, u2^{2} - 129744896\, Q^{13}\, u1\, u2^{2} + 79364096\, Q^{15}\, u1\, u2^{2} - 26869760\, Q^{17}\, u1\, u2^{2} +$
$3145728\, Q^{19}\, u1\, u2^{2} - 3880\, i\, u1^{2}\, u2^{2} + 214272\, i\, Q^{2}\, u1^{2}\, u2^{2} - 4311296\, i\, Q^{4}\, u1^{2}\, u2^{2} + 40286720\, i\, Q^{6}\, u1^{2}\, u2^{2} - 196438016\, i\, Q^{8}\, u1^{2}\, u2^{2} +$
$524476416\, i\, Q^{10}\, u1^{2}\, u2^{2} - 783220736\, i\, Q^{12}\, u1^{2}\, u2^{2} + 657063936\, i\, Q^{14}\, u1^{2}\, u2^{2} - 309329920\, i\, Q^{16}\, u1^{2}\, u2^{2} + 65011712\, i\, Q^{18}\, u1^{2}\, u2^{2} -$
$106240\, Q\, u1^{3}\, u2^{2} + 4291072\, Q^{3}\, u1^{3}\, u2^{2} - 60866560\, Q^{5}\, u1^{3}\, u2^{2} + 410987520\, Q^{7}\, u1^{3}\, u2^{2} - 1465647104\, Q^{9}\, u1^{3}\, u2^{2} + 2881683456\, Q^{11}\, u1^{3}\, u2^{2} -$
$3203268608\, Q^{13}\, u1^{3}\, u2^{2} + 2026110976\, Q^{15}\, u1^{3}\, u2^{2} - 631242752\, Q^{17}\, u1^{3}\, u2^{2} + 33554432\, Q^{19}\, u1^{3}\, u2^{2} - 46560\, i\, u1^{4}\, u2^{2} +$
$2903680\, i\, Q^{2}\, u1^{4}\, u2^{2} - 62765056\, i\, Q^{4}\, u1^{4}\, u2^{2} + 594636288\, i\, Q^{6}\, u1^{4}\, u2^{2} - 2844647424\, i\, Q^{8}\, u1^{4}\, u2^{2} + 7336697856\, i\, Q^{10}\, u1^{4}\, u2^{2} -$
$10593501184\, i\, Q^{12}\, u1^{4}\, u2^{2} + 8597929984\, i\, Q^{14}\, u1^{4}\, u2^{2} - 3594518528\, i\, Q^{16}\, u1^{4}\, u2^{2} + 450887680\, i\, Q^{18}\, u1^{4}\, u2^{2} - 1064960\, Q\, u1^{5}\, u2^{2} +$
$43145216\, Q^{3}\, u1^{5}\, u2^{2} - 6069698$ $340\, Q^{11}\, u1^{5}\, u2^{2} -$
$25783566336\, Q^{13}\, u1^{5}\, u2^{2} + 1403$ $20\, i\, u1^{6}\, u2^{2} +$
$20476928\, i\, Q^{2}\, u1^{6}\, u2^{2} - 445663$ $40281344\, i\, Q^{10}\, u1^{6}\, u2^{2} -$
$57707855872\, i\, Q^{12}\, u1^{6}\, u2^{2} + 40$ $u2^{2} - 5836800\, Q\, u1^{7}\, u2^{2} +$
$231022592\, Q^{3}\, u1^{7}\, u2^{2} - 3113418752\, Q^{5}\, u1^{7}\, u2^{2} + 19156647936\, Q^{7}\, u1^{7}\, u2^{2} - 59719417856\, Q^{9}\, u1^{7}\, u2^{2} + 98927640576\, Q^{11}\, u1^{7}\, u2^{2} -$
$86576726016\, Q^{13}\, u1^{7}\, u2^{2} + 36859543552\, Q^{15}\, u1^{7}\, u2^{2} - 5687476224\, Q^{17}\, u1^{7}\, u2^{2} + 67108864\, Q^{19}\, u1^{7}\, u2^{2} - 1318400\, i\, u1^{8}\, u2^{2} +$
$83460096\, i\, Q^{2}\, u1^{8}\, u2^{2} - 1750466560\, i\, Q^{4}\, u1^{8}\, u2^{2} + 15009816576\, i\, Q^{6}\, u1^{8}\, u2^{2} - 61943316480\, i\, Q^{8}\, u1^{8}\, u2^{2} + 131372548096\, i\, Q^{10}\, u1^{8}\, u2^{2} -$
$144929980416\, i\, Q^{12}\, u1^{8}\, u2^{2} + 81040244736\, i\, Q^{14}\, u1^{8}\, u2^{2} - 19293798400\, i\, Q^{16}\, u1^{8}\, u2^{2} + 838860800\, i\, Q^{18}\, u1^{8}\, u2^{2} - 18841600\, Q\, u1^{9}\, u2^{2} +$
$712146944\, Q^{3}\, u1^{9}\, u2^{2} - 8872656896\, Q^{5}\, u1^{9}\, u2^{2} + 49515855872\, Q^{7}\, u1^{9}\, u2^{2} - 135757168640\, Q^{9}\, u1^{9}\, u2^{2} + 190264115200\, Q^{11}\, u1^{9}\, u2^{2} -$
$135780106240\, Q^{13}\, u1^{9}\, u2^{2} + 45063602176\, Q^{15}\, u1^{9}\, u2^{2} - 4362076160\, Q^{17}\, u1^{9}\, u2^{2} - 3471360\, i\, u1^{10}\, u2^{2} + 202014720\, i\, Q^{2}\, u1^{10}\, u2^{2} -$
$3918004224\, i\, Q^{4}\, u1^{10}\, u2^{2} + 30297948160\, i\, Q^{6}\, u1^{10}\, u2^{2} - 109129236480\, i\, Q^{8}\, u1^{10}\, u2^{2} + 195700981760\, i\, Q^{10}\, u1^{10}\, u2^{2} -$
$175187689472\, i\, Q^{12}\, u1^{10}\, u2^{2} + 76822872064\, i\, Q^{14}\, u1^{10}\, u2^{2} - 12683575296\, i\, Q^{16}\, u1^{10}\, u2^{2} - 35717120\, Q\, u1^{11}\, u2^{2} + 1265500160\, Q^{3}\, u1^{11}\, u2^{2} -$
$13993246720\, Q^{5}\, u1^{11}\, u2^{2} + 67825041408\, Q^{7}\, u1^{11}\, u2^{2} - 156105703424\, Q^{9}\, u1^{11}\, u2^{2} + 174785036288\, Q^{11}\, u1^{11}\, u2^{2} - 97626619904\, Q^{13}\, u1^{11}\, u2^{2} +$
$23018340352\, Q^{15}\, u1^{11}\, u2^{2} - 5611520\, i\, u1^{12}\, u2^{2} + 282558464\, i\, Q^{2}\, u1^{12}\, u2^{2} - 4798283776\, i\, Q^{4}\, u1^{12}\, u2^{2} + 32089702400\, i\, Q^{6}\, u1^{12}\, u2^{2} -$
$95158272000\, i\, Q^{8}\, u1^{12}\, u2^{2} + 134215630848\, i\, Q^{10}\, u1^{12}\, u2^{2} - 91838480384\, i\, Q^{12}\, u1^{12}\, u2^{2} + 27246198784\, i\, Q^{14}\, u1^{12}\, u2^{2} - 36700160\, Q\, u1^{13}\, u2^{2} +$
$1204289536\, Q^{3}\, u1^{13}\, u2^{2} - 11200888832\, Q^{5}\, u1^{13}\, u2^{2} + 43618664448\, Q^{7}\, u1^{13}\, u2^{2} - 77594624000\, Q^{9}\, u1^{13}\, u2^{2} + 61991813120\, Q^{11}\, u1^{13}\, u2^{2} -$
$21139292160\, Q^{13}\, u1^{13}\, u2^{2} - 5079040\, i\, u1^{14}\, u2^{2} + 206831616\, i\, Q^{2}\, u1^{14}\, u2^{2} - 2758279168\, i\, Q^{4}\, u1^{14}\, u2^{2} + 14705229824\, i\, Q^{6}\, u1^{14}\, u2^{2} -$
$32015122432\, i\, Q^{8}\, u1^{14}\, u2^{2} + 28286386176\, i\, Q^{10}\, u1^{14}\, u2^{2} - 10401873920\, i\, Q^{12}\, u1^{14}\, u2^{2} - 15728640\, Q\, u1^{15}\, u2^{2} + 476053504\, Q^{3}\, u1^{15}\, u2^{2} -$
$3422552064\, Q^{5}\, u1^{15}\, u2^{2} + 8405385216\, Q^{7}\, u1^{15}\, u2^{2} - 7784628224\, Q^{9}\, u1^{15}\, u2^{2} + 2952790016\, Q^{11}\, u1^{15}\, u2^{2} - 1966080\, i\, u1^{16}\, u2^{2} +$
$59506688\, i\, Q^{2}\, u1^{16}\, u2^{2} - 427819008\, i\, Q^{4}\, u1^{16}\, u2^{2} + 1050673152\, i\, Q^{6}\, u1^{16}\, u2^{2} - 973078528\, i\, Q^{8}\, u1^{16}\, u2^{2} + 369098752\, i\, Q^{10}\, u1^{16}\, u2^{2} -$

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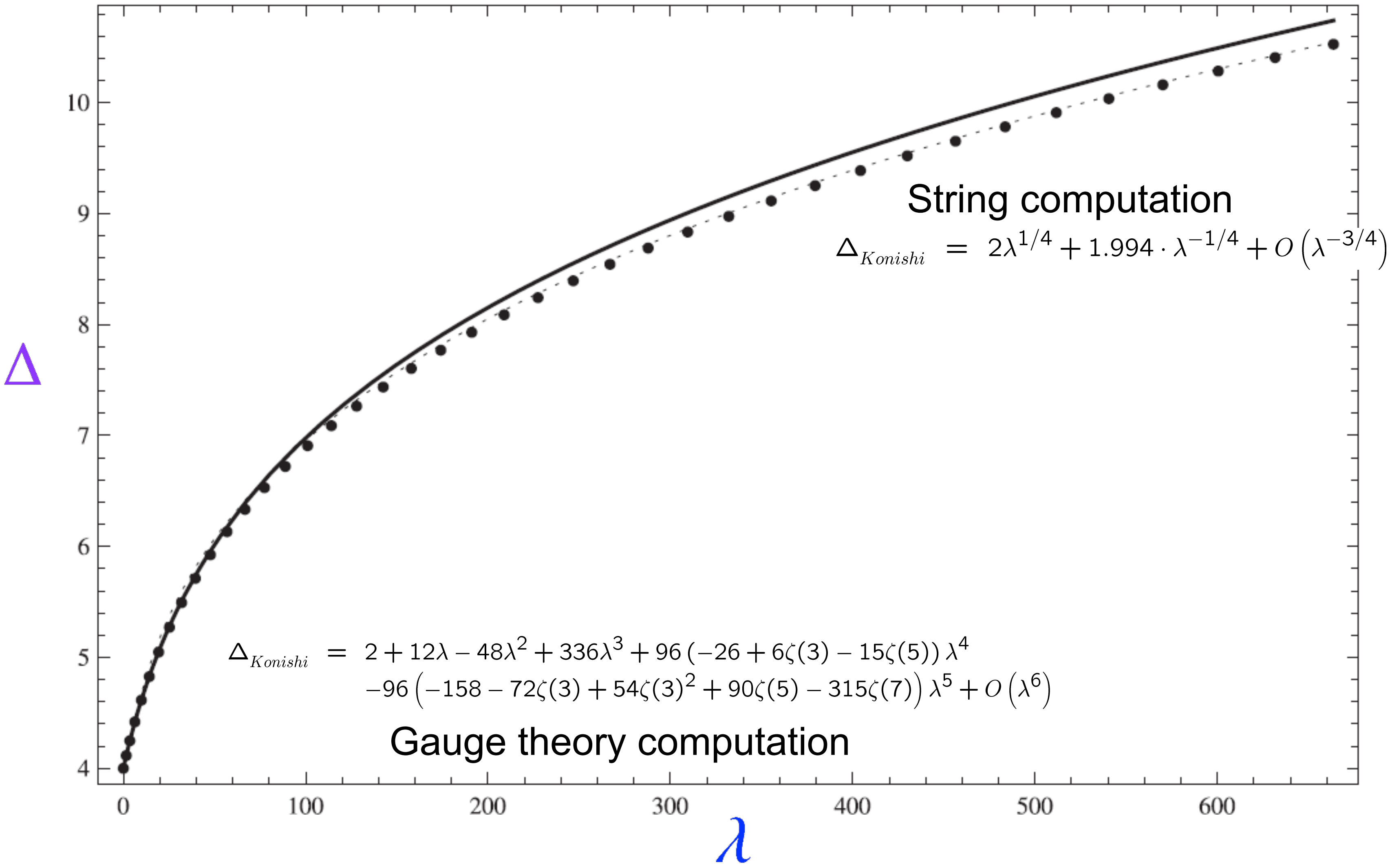
- After sum, we get exactly the discrepancy

$$g^8\Big[-54(1+\delta)^3(-5+3\delta)\zeta(3)-360(1+\delta)^2\zeta(5)$$

$$+\frac{81\,(1-3\,\delta)^2\,(1+\delta)^4}{(1+3\,\delta)^2}\Big]$$

- Exact solution



String computation

$$\Delta_{Konishi} = 2\lambda^{1/4} + 1.994 \cdot \lambda^{-1/4} + O\left(\lambda^{-3/4}\right)$$

$$\begin{aligned}\Delta_{Konishi} = \; & 2 + 12\lambda - 48\lambda^2 + 336\lambda^3 + 96\left(-26 + 6\zeta(3) - 15\zeta(5)\right)\lambda^4 \\ & -96\left(-158 - 72\zeta(3) + 54\zeta(3)^2 + 90\zeta(5) - 315\zeta(7)\right)\lambda^5 + O\left(\lambda^6\right)\end{aligned}$$

Gauge theory computation

Conclusion

perturbation

Non-perturbation

21st century is an era of non-perturbative physics