New Integrable RG Flows with Parafermions

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Based on

Z. Bajnok & CA [arXiv:2407.06582]

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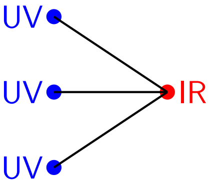
Space 2D QFTs

Can we classify these special 2D QFTs?



Can we classify all possible 2D QFTs which flow into a given IR CFT?

Plan of Talk

1. New Approach [LeClair & CA (2022)]
   - Classify UV CFTs connected to an IR CFT
   - But only UV CFTs can be identified
2. Identifying QFTs [Bajnok & CA (2024)]
   - Focus on the unitary minimal CFTs $\mathcal{M}_p$
   - Find and identify all QFTs which flow into this
3. Concluding remarks

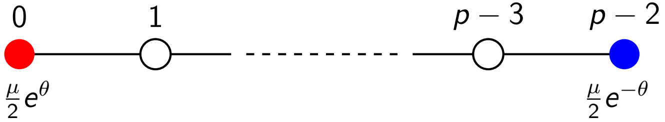
PART 1. RG flows from IR to UV

Conventional Approach (top down)

- UV CFT+a relevant field $\longmapsto$ IR CFT+ irrelevant fields
  - Only a special relevant field maintains the integrability
  - Common in CMP since physics at low T is important to find non-trivial (Wilson-Fisher) fixed points

- (Ex) Zamolodchikov Flows originally based on conjectured TBA

$$\mathcal{M}_{p+1} + \lambda \Phi_{1,3} \to \mathcal{M}_p + \lambda' \Phi_{3,1}, \quad p = 3, 4, \cdots$$



- Many more RG flows have been conjectured
  - have been guessed based on conjectured TBA or NLIE
  - Lagrangians / exact S-matrices are missing

New approach (bottom up) [CA, A.LeClair (2022)]

- IR CFT$+$ irrelevant fields $\longmapsto$ UV CFT$+$a relevant field
- Natural since S-matrices are defined in the IR (infinite volume)
- Common in HEP where UV complete theory is being searched (GUT, SUSY, Superstring, ...)
- Use $T\overline{T}$ as a ladder

$T\overline{T}$ deformations

- Very active developments [Tateo,Zamolodchikov,...]
- Energy-momentum tensor T_2
- All higher conserved charges $= \{ \, [T\overline{T}]_s \, \}$

$$[T\overline{T}]_s = T_{s+1}\overline{T}_{s+1} - \Theta_{s-1}\overline{\Theta}_{s-1}, \quad \partial_{\bar{z}} T_{s+1} = \partial_z \Theta_{s-1}$$

- Preserve integrability
- Exact results possible for even non-integrable theories

Space of 2D IQFTs [Smirnov-Zamolodchikov (2017)]

- Expands the integrable space in infinite dimensions
- If the mother theory is integrable, new integrable QFTs

$$\text{New IQFTs} = \text{an IQFT} + \sum_{s=1}^{\infty} \alpha_s [T\overline{T}]_s$$

- Exact S-matrices are given by additional CDD factors

$$\mathrm{S}(\theta) = \prod_{s=1}^{\infty} e^{2i\alpha_s M^{2s} \sinh(s\theta)} \cdot \mathrm{S}_0(\theta)$$

Swampland (Hagedorn singularity) to cross for UV

- Burgers Equation $(s = 1)$ $[CFT + \alpha_1[T\overline{T}]_1]$

$$\partial_\alpha E + E\partial_R E = 0 \quad \to \quad c_{\text{eff}}(R) = \frac{2c_0}{1 + \sqrt{1 + \frac{2\pi\alpha_1 c_0}{3R^2}}}$$

with square root singularity at $R_c = \sqrt{\frac{2\pi|\alpha_1|c_0}{3}}$

- Singularity occurs also for each $[T\overline{T}]_s$ with $s \geq 1$
- We show that the singularities can be avoided if we fine-tune all α_s

S-matrices of CFTs from a massless limit

- Consider "massive" integrable deformation of a CFT e.g.

$$[G]_k \equiv \frac{G_1 \otimes G_k}{G_{k+1}} + \lambda\, \Phi_{\text{least rel}}$$

- Integrable with exact S-matrix S_0 [Bernard, LeClair, CA (1990)]
- Take $\lambda \to 0^-$ or $M \to 0$ limit
  - rescale the rapidity

$$M \to 0 \;\&\; \theta = \pm\Lambda + \hat{\theta} \quad \text{with finite} \quad M e^{\Lambda} = \mu$$

  - R and L particles appear depending on $\pm$

$$(R) : E = P = \frac{\mu}{2} e^{\hat{\theta}}, \qquad (L) : E = -P = \frac{\mu}{2} e^{-\hat{\theta}}$$

  - S-matrices between RR, LL are the same $\mathsf{S}^{RR}(\hat{\theta}) = \mathsf{S}^{LL}(\hat{\theta}) = \mathsf{S}_0(\hat{\theta})$
  - S-matrices between RL, LR are trivial since $\theta_{12} \to \pm\infty$

S-matrices for deformed CFTs by $T\overline{T}$'s

- CDD factors become trivial in RR, LL sectors

$$\prod_{s=1}^{\infty} e^{2i\alpha_s M^{2s}\sinh(s\theta)} \to 1 \quad \text{as} \quad M \to 0$$

- CDD factors become non-trivial in RL, LR sectors

$$\prod_{s=1}^{\infty} e^{2i\alpha_s M^{2s}\sinh(s\theta)} \to \prod_{s=1}^{\infty} e^{\pm i\alpha_s \mu^{2s} e^{\pm s\hat{\theta}}} \equiv \mathsf{S}_{\mathsf{CDD}}(\hat{\theta})$$

- Summary

$$\boxed{\mathsf{S}^{RR}(\theta) = \mathsf{S}^{LL}(\theta) = \mathsf{S}_0,\ \mathsf{S}^{RL}(\theta) = \mathsf{S}^{LR}(-\theta) = \mathsf{S}_{\mathsf{CDD}}(\theta)}$$

How to restrict S_{CDD}?

- Find all possible S_{CDD} for a given S_0 which satisfy
- UV completeness
  - No Hagedorn singularities
  - c_{eff} in the UV limit is **finite, rational**
- Crossing-Unitarity

$$S_{CDD}(\theta)S_{CDD}(\theta + i\pi) = 1$$

from which S_{CDD} are arbitrary products of only two factors

$$S^{(1)}_{CDD} = -\tanh\left(\frac{\theta-\beta}{2} - \frac{i\pi}{4}\right), \quad S^{(2)}_{CDD} = \frac{\sinh\theta - i\sin\beta}{\sinh\theta + i\sin\beta}$$

- All α_s are fixed exactly in terms of β's
- We can work with UV limit of TBA, namely, “plateaux equations”

Focus on “diagonal” S-matrices theories

- S-matrices for $[G]_1$ have been known [Zamolodchikov,...]
- A CFT can have different S-matrices depending on the choice of the integrable relevant field perturbations
  (ex) Ising CFT by $G = su(2), E_8$
- Plateaux equations

$$x_a^R = \prod_{b=1}^{r} (1 + x_b^R)^{k_{ab}} \prod_{b=1}^{r} (1 + x_b^L)^{\tilde{k}_{ab}}$$

  where exponents are integrals of logarithmic derivatives of S-matrices S_0^{ab} and $\mathrm{S}_{\mathrm{CDD}}^{ab}$ ($\tilde{k}_{ab} = \frac{1}{2}, 1$ or sum of them)
- c_{eff} in terms of some dilog identities

Results: UV CFTs which can flow into Ising CFT $\mathcal{M}_3$

- From the Plateaux equations, we find only

$$c_{\text{eff}} = \frac{7}{10},\ \frac{21}{22},\ \frac{3}{2},\ \frac{15}{2},\ \frac{31}{2}$$

  - "$\frac{7}{10}$" is the Zamolodchikov flow $\mathcal{M}_4 \to \mathcal{M}_3$ ($[su(2)]_2 \to [su(2)]_1$)
  - "$\frac{21}{22}$" is the coset flow $[E_8]_2 \to [E_8]_1$
  - "$\frac{3}{2}$" is the massless SShG model which flows from super-Liouville theory (later) [Kim, Rim, Zamolodchikov,CA (2002)]
  - New: "$\frac{31}{2}$" is supposed to be a flow from E_8 WZW with level 2
  - New: "$\frac{15}{2}$" (?)
- We analyzed other group $G = su(3), su(4)$ etc to classify all possible UV CFTs based on central charges
- But central charges are **not enough** to identify the QFTs

PART 2. Identifying UV complete QFTs

with Z. Bajnok

RSOS (non-diagonal) scattering theory

- Consider $k = p - 2$ with $G = su(2)$

$$\mathcal{M}_p + \lambda \Phi_{1,3}, \qquad \lambda < 0$$

- Particle spectrum: massive kinks

$$a \Big\uparrow b \;=\; K_{ab}(\theta), \qquad a, b = 0, \frac{1}{2} \cdots, \frac{p}{2} - 1, \quad \text{with} \quad |a - b| = \frac{1}{2}$$

- S-matrix of kinks $S^{[\mathrm{p}]}_{\mathrm{RSOS}}(\theta)$: non-diagonal [Bernard, LeClair (1990)]

$$K_{ab}(\theta_1) K_{bc}(\theta_2) \quad \to \quad K_{ad}(\theta_2) K_{dc}(\theta_1)$$

$$\mathsf{S}_p(\theta)^{ab}_{dc} = U(\theta)\, (X^{ac}_{db})^{\frac{i\theta}{2\pi}} \left[(X^{ac}_{db})^{\frac{1}{2}} \sinh\left(\frac{\theta}{p}\right) \delta_{db} + \sinh\left(\frac{i\pi - \theta}{p}\right) \delta_{ac} \right]$$

Massless S-matrices [Fendley, Saleur, Zamolodchikov ('93)]

- Massless limit $\lambda \to 0^-$

$$\mathsf{S}_p^{RR}(\theta) = \mathsf{S}_p^{LL}(\theta) = \mathsf{S}_p(\theta)$$

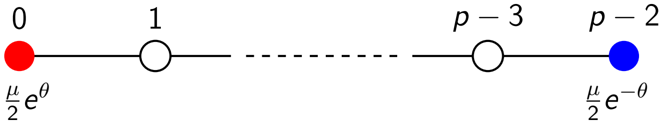
- YBE and Crossing-Unitarity relations determine

$$\mathsf{S}_p^{RL}(\theta) = \mathsf{S}_p^{LR}(-\theta) \propto \mathsf{S}_p\left(\theta - \frac{ip\pi}{2}\right)$$

for the S-matrices of the IR theory $\mathcal{M}_p + \lambda' \Phi_{3,1}$

- Derived the conjectured TBA (partially) shown before

$$\epsilon_a(\theta) = \frac{\mu R}{2}(\delta_{a0} e^{\theta} + \delta_{a,p-2} e^{-\theta}) - \sum_{b=0}^{p-2} \mathbb{I}_{ab} \varphi \star \log[1 + e^{-\epsilon_b}](\theta)$$



TBA for $[T\overline{T}]_s$ deformed minimal CFT $\mathcal{M}_p$

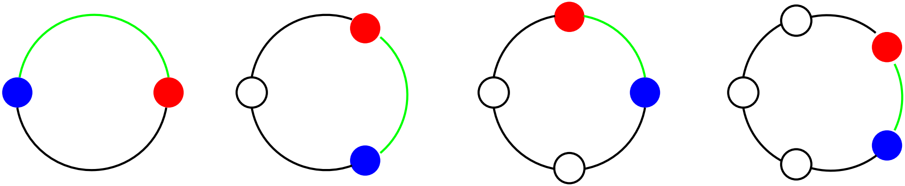
- $[T\overline{T}]_s$ deformations

$$\mathcal{M}_p + \lambda' \, \Phi_{3,1} + \sum_{s \geq 1} \alpha_s \, [T\overline{T}]_s$$

- Introduce CDD factors to RL, LR sectors

$$\tilde{\mathsf{S}}_p^{RL}(\theta) = \mathsf{S}_{\mathrm{CDD}} \cdot \mathsf{S}_p^{RL}(\theta)$$

- Being diagonal, $\mathsf{S}_{\mathrm{CDD}}$ introduces additional kernels between R & L



UV complete theories

- Plateaux equations

$$
\begin{aligned}
x_n &= (1+x_{n-1})^{1/2}(1+x_{n+1})^{1/2}, \quad n = 1, \cdots, p-3, \\
x_0 &= (1+x_1)^{1/2}(1+x_{p-2})^{\tilde{k}}, \quad x_{p-2} = (1+x_{p-3})^{1/2}(1+x_0)^{\tilde{k}}
\end{aligned}
$$

- UV complete only with $\tilde{k} = \frac{1}{2}$ which means

$$
\mathrm{S}^{(1)}_{\mathrm{CDD}} = -\tanh\left(\frac{\theta-\beta}{2} - \frac{i\pi}{4}\right) \quad \rightarrow \quad \varphi_{RL} = \frac{1}{\cosh(\theta-\beta)}
$$

- We also notice that only real β ($p > 3$) becomes UV complete and

$$
c_{\text{eff}} = 3\,\frac{p-1}{p+1}
$$

For $p = 3$, massless Super sinh-Gordon model

- sinh-Gordon model with $N = 1$ super [Kim, Rim, Zamolodchikov, CA (2002)]

$$\mathcal{L} = \mathrm{Kin.} - \frac{i}{2}\psi\overline{\psi}W''(\phi) + 2\pi[W'(\phi)]^2, \quad W(\phi) = -\mu \sinh(b\phi)$$

- Supersymmetry is spontaneous broken

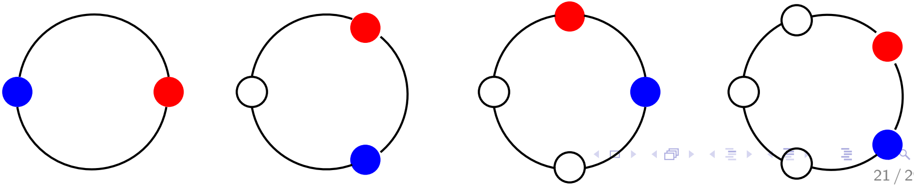
$$V(\phi) \propto [W'(\phi)]^2 = \cosh^2(b\phi) > 0$$

- massless Goldstino fermion is only spectrum in the IR limit (Ising model)
- TBA is given by

$$\begin{aligned}
\epsilon^R(\theta) &= \frac{\mu R}{2} e^{\theta} - \varphi^{RL} \star \log[1 + e^{-\epsilon^L}], \\
\varphi^{RL}(\theta) &= \frac{1}{\cosh(\theta - ia)} + \frac{1}{\cosh(\theta + ia)}, \quad a = \frac{1 - b^2}{1 + b^2}
\end{aligned}$$

Two theories match when $\beta = 0$ and $a = 0$

- Ising model $\mathcal{M}_{p=3}$ deformed by $[T\overline{T}]_s$ with $(\beta = 0)$ $=$ SshG model at self-dual coupling
- Conjecture: $\mathcal{M}_{p\geq 3}$ deformed by $[T\overline{T}]_s$ with $(\beta = 0)$ $= \mathbb{Z}_{p-1}$ Parafermionic shG model at self-dual coupling



Parafermionic shG model

- PshG model

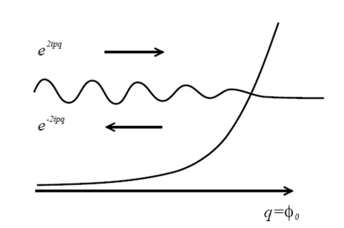
$$\mathcal{L}_{\mathsf{PShG}} = \mathcal{L}_{\mathsf{PF}} + \frac{1}{4\pi}(\partial_\mu \phi)^2 - \kappa \left(\psi_1 \overline{\psi}_1 e^{2b\phi} + \eta\, \psi_1^\dagger \overline{\psi}_1^\dagger e^{-2b\phi} \right) + \cdots$$

- ψ_1 is a $\mathbb{Z}_k$ PF with $\Delta = 1 - \frac{1}{k}$ ($k \equiv p - 1$)
- Fractional SUSY
  - $\eta = 0$ gives Parafermionic LFT [Baseilhac, Fateev (1998)]
  - $\eta = 1$ massive phase
  - $\eta = -1$ massless phase where FSUSY is spontaneously broken
- These types of QFTs compute c_{eff} from momentum quantization using Reflection amplitudes

Reflection amplitudes of LFTs

- Primary fields $e^{2\alpha\phi}$ and $e^{2(Q-\alpha)\phi}$ identified upto constant

$$e^{2(\frac{Q}{2}+ip)\phi} = \mathcal{R}(p)e^{2(\frac{Q}{2}-ip)\phi}, \qquad \alpha = \frac{Q}{2}+ip,\ Q = b+\frac{1}{kb}$$

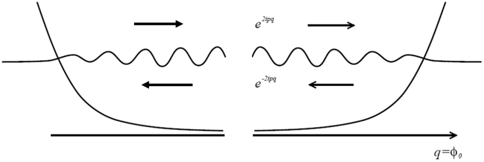


- This amplitude has been computed in the PLFT [Baseilhac, Fateev (1998)]

Momentum quantization condition

- the perturbation introduces another wall

$$\delta^{(k)}(p) = \pi + 4Qp \ln \frac{R}{2\pi} = \sum_{n=\mathrm{odd}} \delta_n p^n, \quad \mathcal{R}(p) = e^{i\delta^{(k)}(p)}$$



- Scaling function can be obtained from

$$\begin{aligned} c_{\mathrm{eff}} &= \frac{3k}{k+2} - 24p^2 + \mathcal{O}(R) \\ &= \frac{3k}{k+2} - \frac{3\pi^2}{2Q^2 \ln^2 x} - \frac{3\pi^2\delta_1}{4Q^3 \ln^3 x} - \frac{9\pi^2\delta_1^2}{32Q^4 \ln^4 x} - \frac{3(2\pi^2\delta_1^3 + \pi^4\delta_3)}{64Q^5 \ln^5 x} + \ldots \quad (x = \tfrac{R}{2\pi}) \end{aligned}$$

Reflection amplitude vs. massless TBA

- Need to solve numerically with high precision

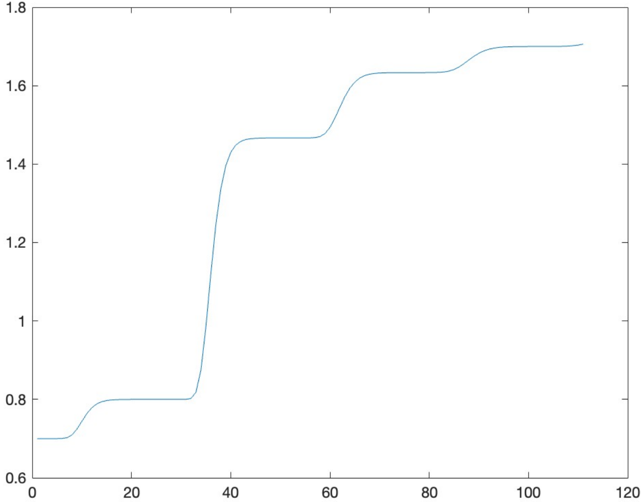
$$c_{\text{eff}}(r) = \frac{3k}{k+2} + \sum_{n=2} \frac{c_n(k)}{(\log r)^n} + O(r)$$

- Two match very well at self-dual coupling $b = \frac{1}{\sqrt{k}}$

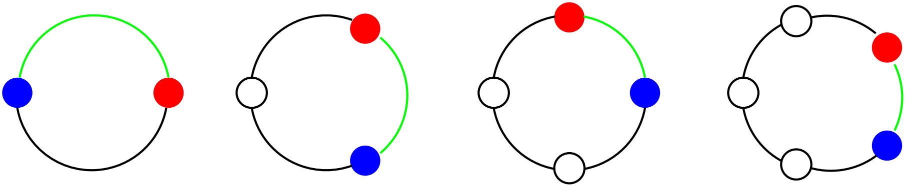
	c_2	c_3	c_4	c_5
$k=2$	7.402199	42.7620	185.218	714.563
	7.402203	42.7628	185.282	717.247
$k=3$	11.10332	77.8573	409.598	1924.84
	11.10330	77.8543	409.425	1919.36
$k=4$	14.8045	119.428	722.797	3898.75
	14.8044	119.4197	722.475	3892.55

What if $\beta \neq 0$ for generic $p \geq 3$?

- Numerical solutions of the massless TBA show "Roaming" [$p = 4$, $\beta = 10\pi$]



Roaming TBAs with $\beta \gg 1$



- Each plateaux corresponds to

$$\mathcal{M}_{k+1} \to \mathbb{Z}_k\mathcal{M}_1 \to \mathbb{Z}_k\mathcal{M}_{k+1} \to \mathbb{Z}_k\mathcal{M}_{2k+1}$$

where

$$\mathbb{Z}_k\mathcal{M}_\ell = \frac{su(2)_k \otimes su(2)_\ell}{su(2)_{k+\ell}}$$

- Reproduce PF staircase TBA conjectured by [Dorey, Ravanini (1993)]

Concluding Remarks

- We have found exact massless S-matrices from which we have derived TBA for new RG flows
- We have found that there is only one new RG flow to $\mathcal{M}_p$ for $p > 3$ if the CFT is in RSOS (can not exclude existence of other basis)
- and we have identified those as PF sinh-Gordon ($\beta = 0$) and PF Roaming ($\beta \gg 1$) models
- Can we find exact S-matrices in this way for all conjectured TBA and NLIE?
- Can we generalize this approach to other irrelevant deformations than $T\overline{T}$'s?
- Connection with non-invertible symmetries

Thank you for attention!