NLIE of Sausage model

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Based on 1701.08933 (J. Phys. A)

Work with
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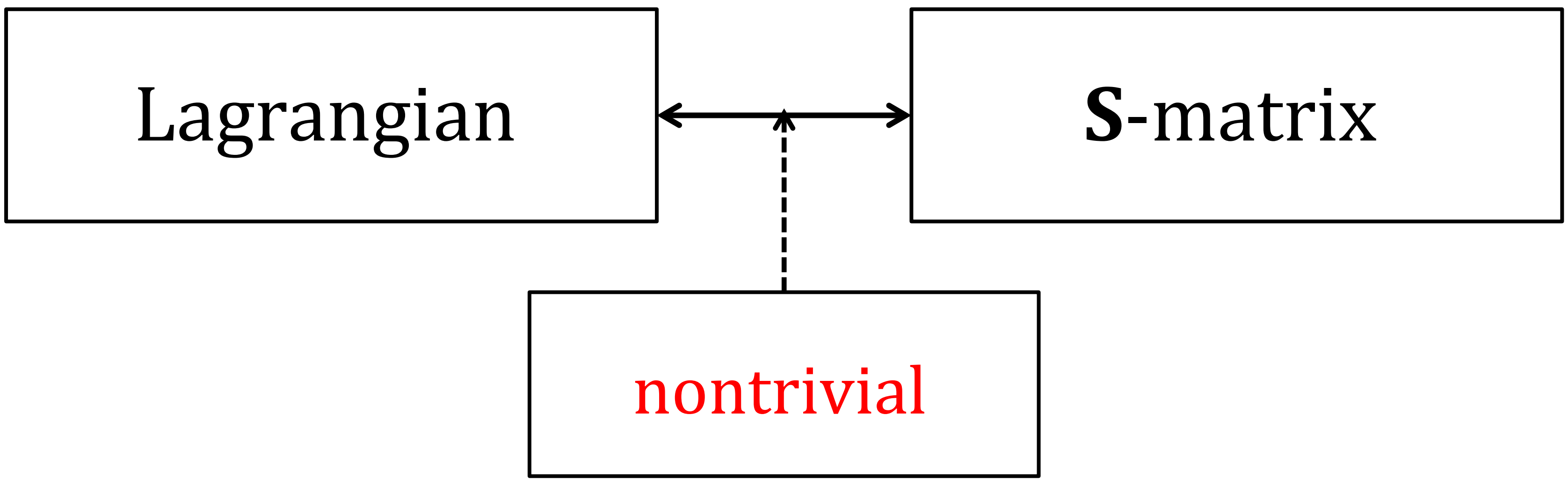
S-matrix program

- Integrable QFT→Exact S-matrix→TBA
  - (ref) Review lecture by Z. Bajnok

- Main Goal: Find new integrable QFTs
  - One- or two-parameter extensions of known integrable QFTs
  - (ex) Lattice analogy: XXX → XXZ → XYZ

R vs. S

R	**S**
YBE	YBE
Define model Hamiltonian	Result of the int. QFT

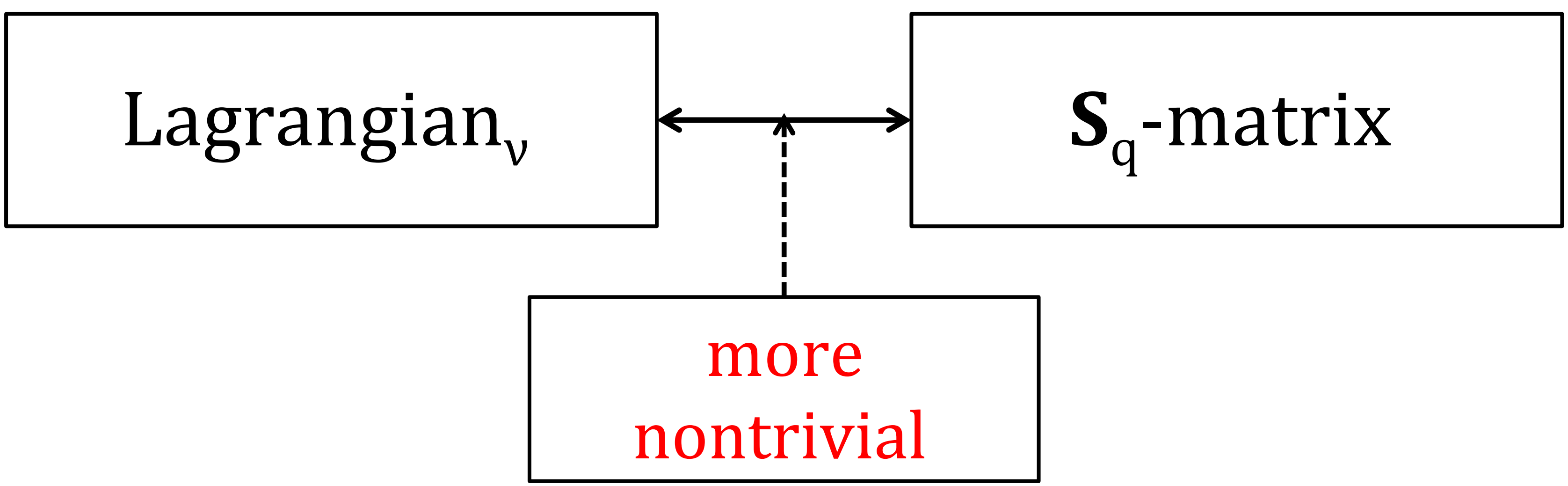


“Zoo” of Integrable QFTs

classes	(ex)	$\mathcal{L}$	spectrum	**S**
Affine Toda theories	Sinh-Gordon	✓	fund. fields	diago.
NL σ models	O(3), AdS/CFT	✓	Reps	non-diago.
Perturbed CFTs	RSOS	✗	kinks	non-diago.

Integrable deformations

- Deform S-matrix
  - Drinfeld-Reshetikhin twist
  - Quantum Group
- Deform Lagrangian
  - Classical Yang-Baxter algebra [Klimsik]
  - Discrete symmetries of target mainfolds



Ex. deformed AdS/CFT

String manifolds

γ-deform: TsT

η-deform: class. YBE

SYM theories

$\mathcal{L} = \ldots + [\Phi,\Phi]_\beta^2$

Non-Comm. space-time

S-matrix

$F_\beta S_{su(2|2)} F_\beta$

$q\text{-}S_{su(2|2)}$

simpler example

O(3) σ-model → sausage σ-model

Outline

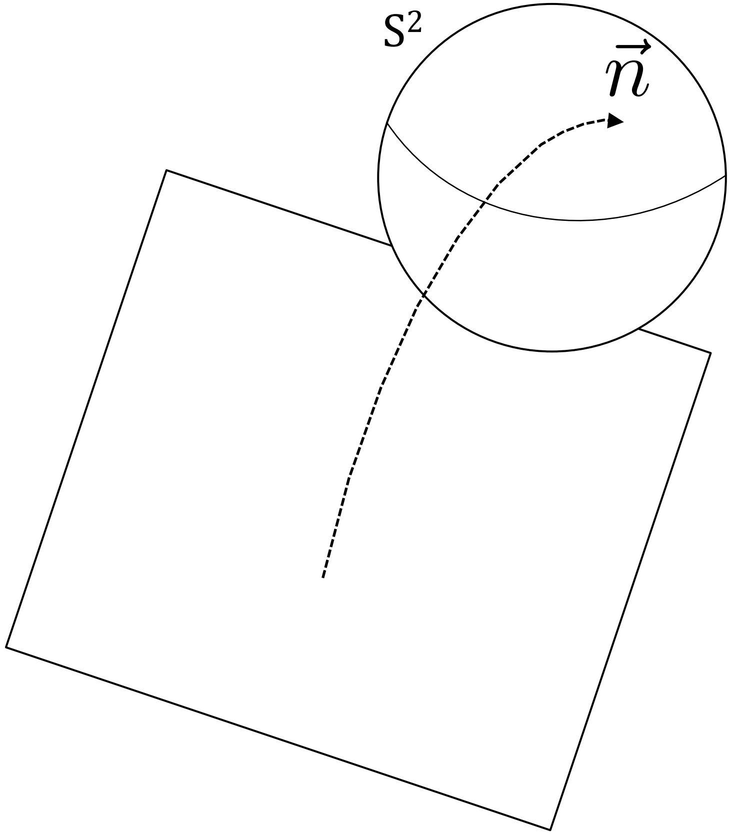
1. Review of Sausage σ-model

2. Derive non-liner integral equation

3. UV and IR limits

4. Match Lagrangian with S-matrix

O(3) σ-model



$$|\vec{n}| = 1$$

$$\mathcal{A} = \frac{1}{2g}\int d^2x\,(\partial_\mu \vec{n})^2 + i\vartheta T$$
$$T = \frac{1}{8\pi}\epsilon^{\mu\nu}\vec{n}\cdot(\partial_\mu \vec{n}\times\partial_\nu \vec{n})$$

Haldane Conjecture: O(3) SM is equivalent to spin-s anti-ferro Heisenberg models in large s limit

s=integer ←→ $\vartheta = 0$

s=half-integer ←→ $\vartheta = \pi$

S-matrix of O(3) σ-model

- Integrable for $\vartheta=0, \pi$ [Zamolodchikov[2]]

$$\vartheta = 0 : \text{massive triplet } (+,0,-)\ m \sim e^{-\pi/g}$$

$$S(\theta) = \frac{\theta + 2\pi i}{\theta - 2\pi i}\mathbf{P}_0 + \frac{\theta + 2\pi i}{\theta - 2\pi i}\frac{\theta - \pi i}{\theta + \pi i}\mathbf{P}_1 + \frac{\theta - \pi i}{\theta + \pi i}\mathbf{P}_2$$

$$\vartheta = \pi : \text{massless L}-, \text{R}-\text{doublet}$$

$$S_{LL}(\theta) = S_{RR}(\theta) = S_{LR}(\theta) = \frac{\Gamma\left(\frac{1}{2}+\frac{\theta}{2\pi i}\right)\Gamma\left(-\frac{\theta}{2\pi i}\right)}{\Gamma\left(\frac{1}{2}-\frac{\theta}{2\pi i}\right)\Gamma\left(\frac{\theta}{2\pi i}\right)}\frac{\theta\mathbf{1} - i\pi \mathsf{P}}{\theta - i\pi}$$

- $\vartheta=\pi$: RG flow to IR CFT = $su(2)_1$ WZW

Sausage SM: defined by S-matrix $SST^{(\pm)}_\lambda$

Fateev, Onofri, Zamolodchikov (1993)

0<λ<1/2: repulsive

(λ>1/2: very complicated)

$SST_\lambda^{(+)}$: massive triplet $(+,0,-)$ $m \sim e^{-\pi/g}$

$$S_{++}^{++}(\theta) = S_{+-}^{+-}(i\pi - \theta) = \frac{\sinh(\lambda(\theta - i\pi))}{\sinh(\lambda(\theta + i\pi))},$$

$$S_{+0}^{0+}(\theta) = S_{+-}^{00}(i\pi - \theta) = \frac{-i\sin(2\pi\lambda)}{\sinh(\lambda(\theta - 2i\pi))} S_{++}^{++}(\theta),$$

$$S_{+0}^{+0}(\theta) = \frac{\sinh(\lambda\theta)}{\sinh(\lambda(\theta - 2i\pi))} S_{++}^{++}(\theta),$$

$$S_{+-}^{-+}(\theta) = -\frac{\sin(\pi\lambda)\sin(2\pi\lambda)}{\sinh(\lambda(\theta - 2i\pi))\sinh(\lambda(\theta + i\pi))}, \quad S_{00}^{00}(\theta) = S_{+0}^{+0}(\theta) + S_{-+}^{+-}(\theta)$$

$SST_\lambda^{(-)}$: massless doublet

Quantum group deformation of O(3) S-matrices

$$U_{++}^{++}(\theta) = U_{--}^{--}(\theta) = U_0(\theta),$$

$$U_{+-}^{+-}(\theta) = U_{-+}^{-+}(\theta) = -\frac{\sinh(\lambda\theta/(1-\lambda))}{\sinh(\lambda(\theta - i\pi)/(1-\lambda))} U_0(\theta),$$

$$U_{-+}^{+-}(\theta) = U_{+-}^{-+}(\theta) = -i\frac{\sin(\pi\lambda/(1-\lambda))}{\sinh(\lambda(\theta - i\pi)/(1-\lambda))} U_0(\theta),$$

$$U_0(\theta) = -\exp\left[i\int_0^\infty \frac{\sinh((1-2\lambda)\pi\omega/(2\lambda))\sin(\omega\theta)}{\cosh(\pi\omega/2)\sinh((1-\lambda)\pi\omega/(2\lambda))}\frac{d\omega}{\omega}\right]$$

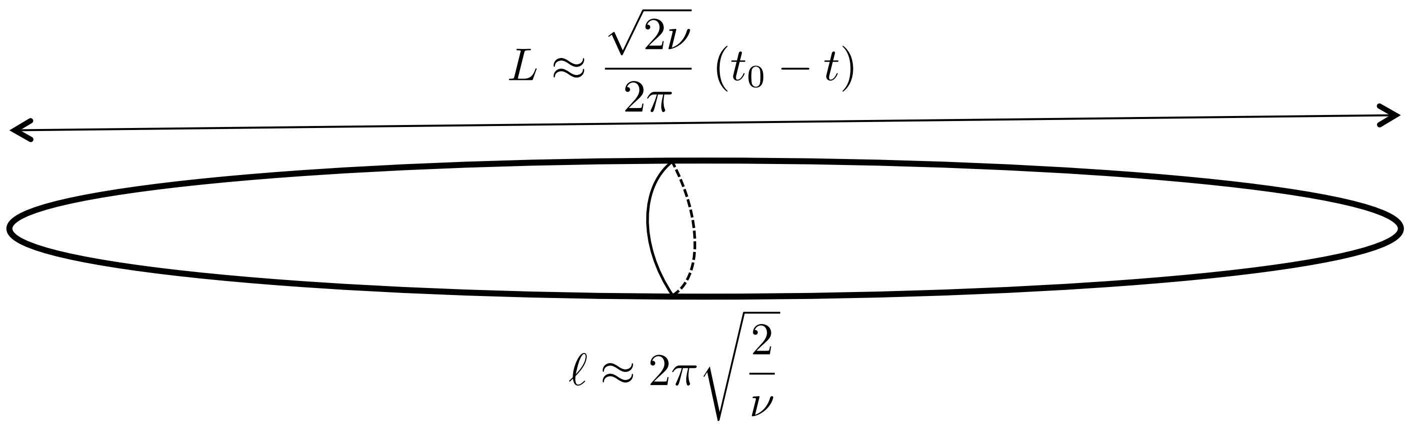
Effective Lagrangian of SSM

Fateev, Onofri, Zamolodchikov (1993)

RG analysis for near UV limit ($\nu \to 0$, $t \to -\infty$)

$$\mathcal{A}[\mathrm{SSM}_\nu^{(\vartheta)}] = \frac{1}{2g(t)} \int d^2x \frac{(\partial_\mu \vec{n})^2}{1 - \frac{\nu^2 n_3^2}{2g(t)^2}} + i\vartheta T$$

$$g(t) = \frac{\nu}{2} \coth \frac{\nu(t_0 - t)}{4\pi}$$



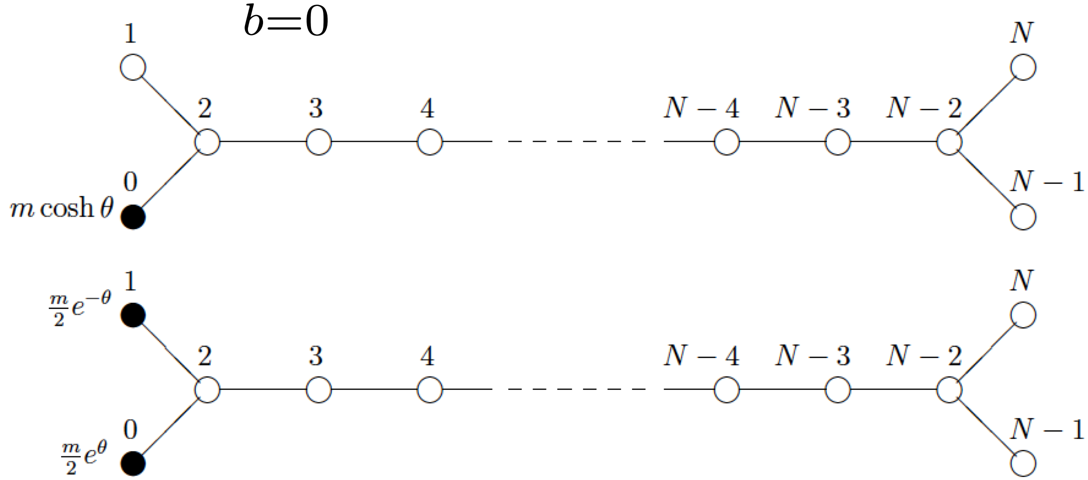
- Classical integrability: Bazhanov, Kotousov, Lukyanov 1706.09941

TBA and Y-system of $SST^{(\pm)}_{\lambda}$

- Derived only for

$$\lambda = \frac{1}{N},\ \ N = 2, 3, \ldots$$

$$\epsilon_a(\theta) = R\, \mathsf{e}_a(\theta) - \sum_{b=0}^{N} \phi_{ab} \star \log(1 + e^{-\epsilon_b})(\theta), \qquad \phi_{ab}(\theta) = \frac{\ell_{ab}}{\cosh\theta}$$



- Y-system ("D"-type)

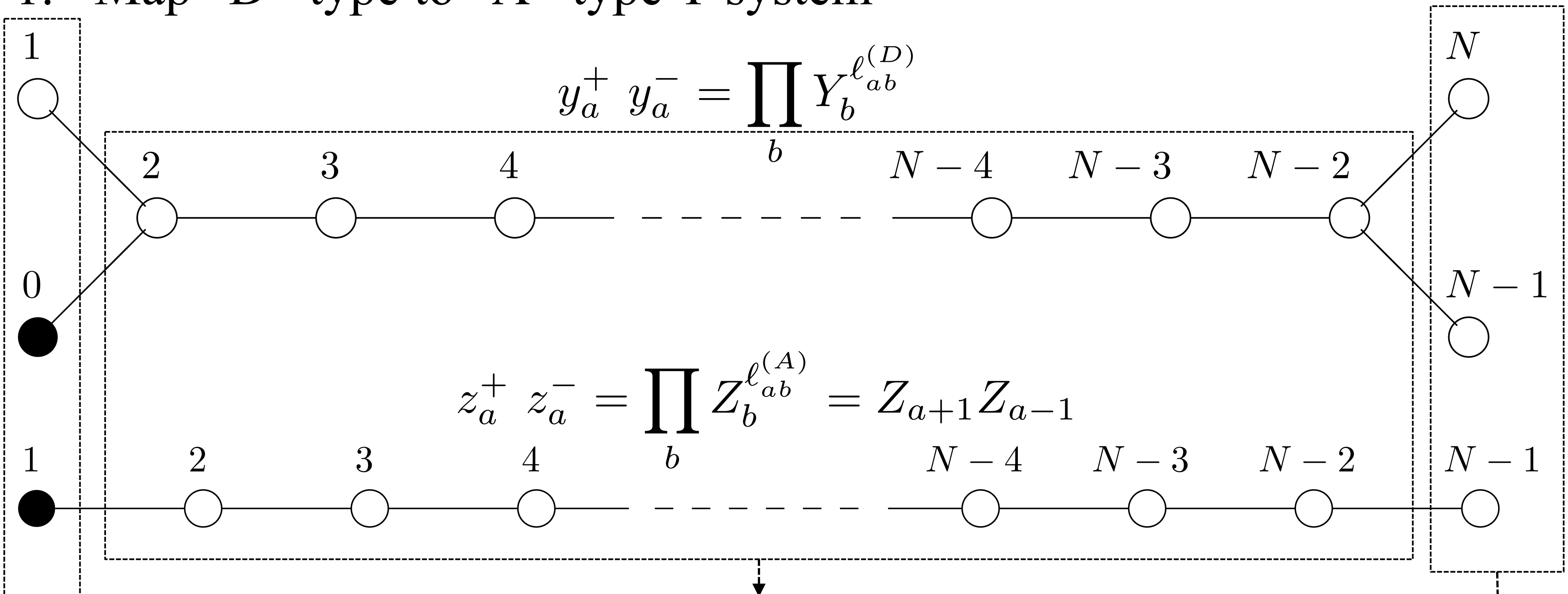
$$y_a^+\ y_a^- = \prod_b Y_b^{\ell^{(D)}_{ab}}, \qquad y^{\pm} = y(\theta \pm \frac{i\pi}{2})$$

TBA vs. NLIE

- TBA: [Al.B. Zamolodchikov]
  - Direct relations to spectrum and S-matrix
  - Many (even infinite) coupled integral equations
- NLIE: [Klumper,Pearce;Destri,deVega;Ravanini et.al; Dunning,Suzuki,…]
  - Simpler, valid for generic coupling, …
  - Need lattice formulation
  - Not directly related to QFT
- algebraic method even without lattice formulation [Balog, Hegedus, …]
  - TBA (Y-system) → T-system → T-Q system → NLIE

Derivation of NLIE for any λ

1. Map "D"-type to "A"-type Y-system



$$y_a^+ \; y_a^- = \prod_b Y_b^{\ell_{ab}^{(D)}}$$

$$z_a^+ \; z_a^- = \prod_b Z_b^{\ell_{ab}^{(A)}} = Z_{a+1} Z_{a-1}$$

$$z_k^+ z_k^- \quad = \quad Z_{k-1} Z_{k+1}, \qquad z_k \equiv y_k, \qquad k = 2, \ldots, N-2,$$

$$Z_1 \quad = \quad Y_0 Y_1, \qquad Z_{N-1} = Y_{N-1} Y_N$$

2. T-system

$$T_k^+ T_k^- = 1 + T_{k-1} T_{k+1}, \qquad k = 2, \ldots, N-2$$

$$z_k = T_{k-1} T_{k+1}, \qquad T_k^+ T_k^- = Z_k, \qquad k = 1, \ldots, N-1$$

with extra condition

$$Z_{N-2} = T_{N-2}^+ T_{N-2}^- = Y_{N-2} = y_N^+ y_N^- = y_{N-1}^+ y_{N-1}^- \quad \rightarrow \quad y_N = y_{N-1} = T_{N-2}$$

$$Z_{N-1} = T_{N-1}^+ T_{N-1}^- = Y_{N-1} Y_N = (1 + T_{N-2})^2 = 1 + T_{N-2} T_N \quad \rightarrow \quad \boxed{T_N = 2 + T_{N-2}}$$

3. T-Q system

$$f^{[k]}(\theta) \equiv f(\theta + \frac{i\pi k}{2})$$

$$T_{k+1} Q^{[k]} - T_k^- Q^{[k+2]} = \bar{Q}^{[-k-2]}, \qquad T_k^- \bar{Q}^{[-k]} - T_{k-1} \bar{Q}^{[-k-2]} = Q^{[k]}$$

with $\boxed{\bar{Q} = Q^{[2N]}}$

4. From T-Q to NLIE

$$b_k = \frac{Q^{[k+2]} T_k^-}{\bar{Q}^{[-k-2]}}, \qquad B_k = 1 + b_k = \frac{Q^{[k]} T_{k+1}}{\bar{Q}^{[-k-2]}}$$

Fourier transform of logarithmic derivatives $\widetilde{f^{[\alpha]}}(\omega) = p^\alpha \tilde{f}(\omega), \qquad p \equiv e^{\frac{\omega\pi}{2}}$

$$\begin{aligned} \tilde{b}_k &= p^{k+2}\tilde{Q} + p^{-1}\tilde{T}_k - p^{-k-2}\tilde{\bar{Q}}, \\ \tilde{B}_k &= p^k \tilde{Q} + \tilde{T}_{k+1} - p^{-k-2}\tilde{\bar{Q}} \end{aligned}$$

Eliminate Q using $\tilde{\bar{Q}} = p^{2N}\tilde{Q}$

Analytically continue: N→1/λ

$$\begin{aligned} \tilde{b} &= \tilde{K}\,(\tilde{B} - \tilde{\bar{B}}) + p^{-1}\tilde{s}\tilde{Y}_1\tilde{Y}_0, \\ \tilde{\bar{b}} &= \tilde{K}\,(\tilde{\bar{B}} - \tilde{B}) + p\tilde{s}\tilde{Y}_1\tilde{Y}_0, \\ \tilde{y} &= p\tilde{s}\tilde{B} + p^{-1}\tilde{s}\tilde{\bar{B}} \end{aligned}$$

$$\tilde{s} = \frac{1}{p+p^{-1}} = \frac{1}{2\cosh\frac{\omega\pi}{2}}$$

$$\tilde{K} = \frac{\sinh\left(\frac{\omega\pi(1-3\lambda)}{2\lambda}\right)}{2\sinh\left(\frac{\omega\pi(1-2\lambda)}{2\lambda}\right)\cosh\frac{\omega\pi}{2}}$$

NLIE in rapidity space

SST$^{(+)}$

$$\log a = K \star \log(1+a) - K^{[2\alpha]} \star \log(1+\bar{a}) + s^{[\alpha-1]} \star [\log(1+y) + \log(1+\xi y)]\,,$$
$$\log \bar{a} = K \star \log(1+\bar{a}) - K^{[-2\alpha]} \star \log(1+a) + s^{[1-\alpha]} \star [\log(1+y) + \log(1+\xi y)]\,,$$
$$\log y = s^{[1-\alpha]} \star \log(1+a) + s^{[\alpha-1]} \star \log(1+\bar{a})$$

SST$^{(-)}$

$$\log a = K \star \log(1+a) - K^{[2\alpha]} \star \log(1+\bar{a}) + s^{[\alpha-1]} \star \left[\log(1+\xi^+ y) + \log(1+\xi^- y)\right],$$
$$\log \bar{a} = K \star \log(1+\bar{a}) - K^{[-2\alpha]} \star \log(1+a) + s^{[1-\alpha]} \star \left[\log(1+\xi^+ y) + \log(1+\xi^- y)\right],$$
$$\log y = s^{[1-\alpha]} \star \log(1+a) + s^{[\alpha-1]} \star \log(1+\bar{a})$$

Vacuum energy

$$E(r) = -\frac{m}{2\pi} \int_{-\infty}^{\infty} \cosh\theta \log(1+\xi y), \quad \xi = e^{-mr\cosh\theta}$$
$$E(r) = -\frac{m}{4\pi} \int_{-\infty}^{\infty} \left[e^{\theta} \log(1+\xi^+ y) + e^{-\theta} \log(1+\xi^- y)\right], \quad \xi^{\pm} = e^{-mr\exp(\pm\theta)/2}$$

IR limit of SST$^{(+)}$

- Linearized NLIE (ξ<<1)

$$a = z(1+w+\dots), \quad y = h(1+u+\dots), \qquad z=2,\ h=3$$

$$w = \frac{2}{3}K\star w - \frac{2}{3}K^{[2\alpha]}\star\bar{w} + s^{[\alpha-1]}\star\left(3\xi+\frac{3}{4}u\right),$$

$$\bar{w} = \frac{2}{3}K\star \bar{w} - \frac{2}{3}K^{[2\alpha]}\star w + s^{[1-\alpha]}\star\left(3\xi+\frac{3}{4}u\right),$$

$$u = \frac{2}{3}s^{[1-\alpha]}\star w + \frac{2}{3}s^{[\alpha-1]}\star\bar{w}$$

- Solutions by F.T.

$$\tilde{u} = \frac{1}{3}\tilde{\xi}\tilde{\varphi}, \qquad \tilde{\varphi} = 8\ \frac{\sinh\left[\pi\omega\left(\frac{1}{2\lambda}-1\right)\right]}{\sinh\frac{\pi\omega}{2\lambda}} - 4\ \frac{\sinh\left[\pi\omega\left(\frac{1}{2\lambda}-2\right)\right]}{\sinh\frac{\pi\omega}{2\lambda}}$$

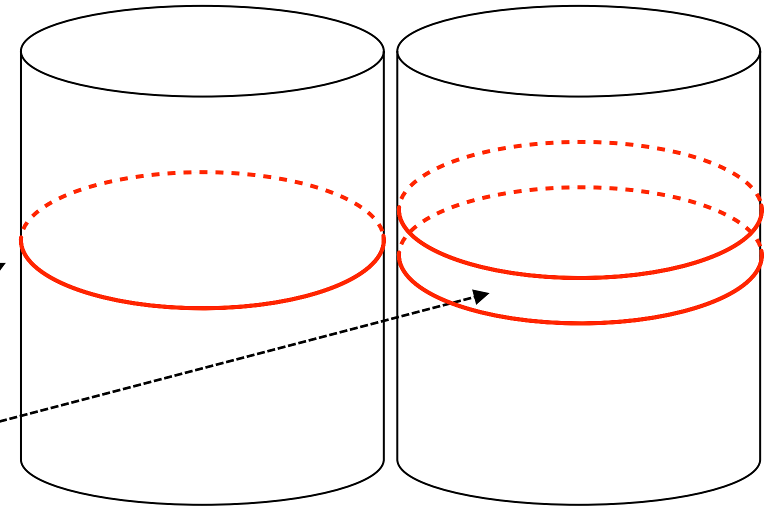
• Virial expansion

$$E = E^{(1)} + E_1^{(2)} + E_2^{(2)} + \mathcal{O}(e^{-3mr}),$$

$$E^{(1)} = -\frac{e_1 m}{2\pi}\int_{-\infty}^{\infty} \cosh\theta\ e^{-mr\cosh\theta}\ d\theta,$$

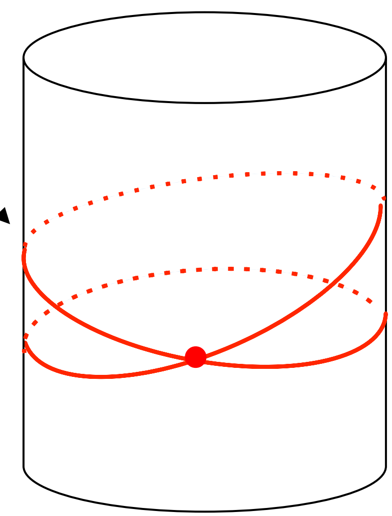
$$E_1^{(2)} = \frac{e_2 m}{4\pi}\int_{-\infty}^{\infty} \cosh\theta\ e^{-2mr\cosh\theta}\ d\theta,$$

$$E_2^{(2)} = -\frac{m}{2\pi}\int_{-\infty}^{\infty} d\theta \cosh\theta\ e^{-mr\cosh\theta}\ \int_{-\infty}^{\infty} d\theta' \varphi(\theta-\theta')\ e^{-mr\cosh\theta'}\ d\theta'$$



$$e_1 = 3, \quad e_2 = 9$$

$$\varphi(\theta) = \frac{1}{2\pi i}\frac{d}{d\theta}\log\det S_\lambda^{(+)}(\theta)$$



UV limit

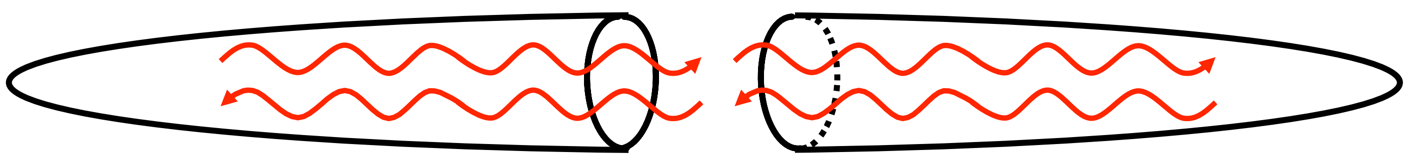
- NLIE as infinite order DE [Al.B.Zamolodchikov]

$$\delta_{a0}\, mr \cosh\theta = \varepsilon_a(\theta) + \sum_b \sum_{n=0}^{\infty} \tilde{\Psi}_{ab,n} L_b^{(n)}(\theta)$$

$$c(r) \approx 2 - \frac{3\pi^2}{2} \frac{\lambda^{-1} - 2}{(\log(mr) + C)^2} + \dots$$

- Zero-mode dynamics from Lagrangian [FOZ]

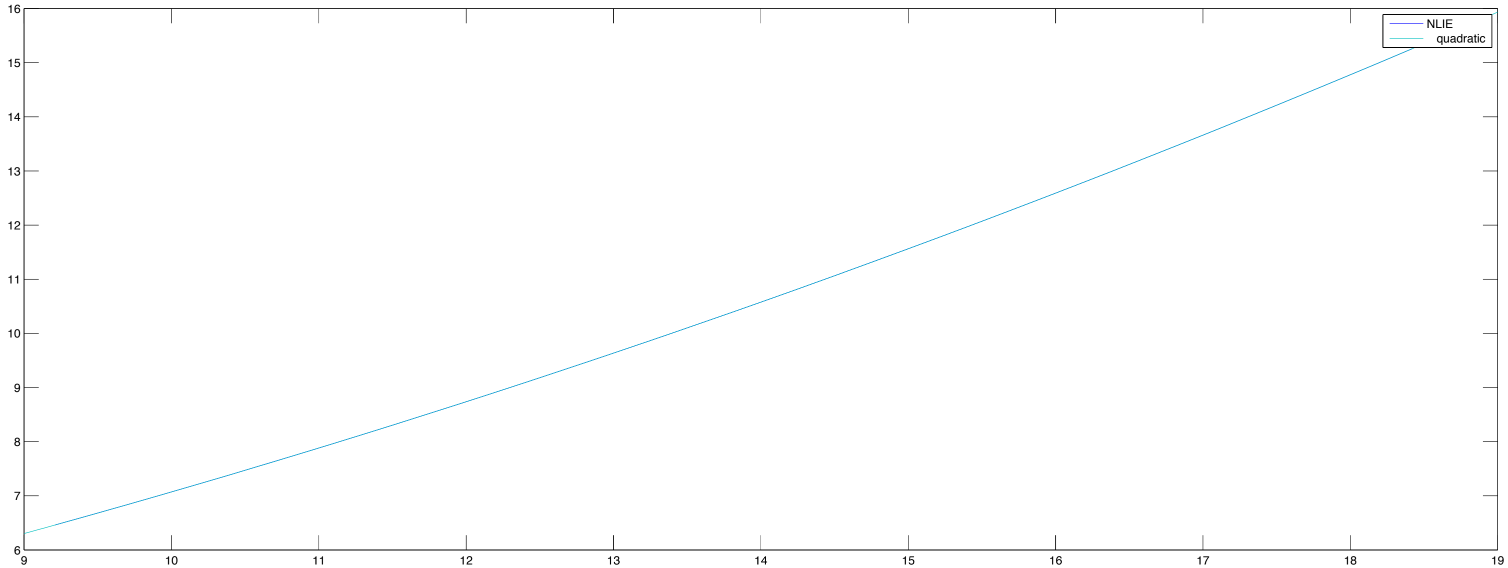
"Cigar" or sine-Liouville



$$c(r) = 2 - \frac{4\pi}{\nu} \frac{3\pi^2}{2(\log(mr) + \delta)^2}$$

Parametric relation: L vs. S

$$\frac{\nu}{4\pi} = \frac{\lambda}{1 - 2\lambda}$$



Summary and Discussion

- Derived NLIE for any λ (<1/2) from TBA by taking analytic continuation
- Satisfy all consistent checks
- Parametric relation between ν and λ

- Bazhanov, Kotousov, Lukyanov 1706.09941 proposed another NLIE which looks similar but not exactly same.
  - But generates same numerics
  - Need to be clarified
- Attractive regime (λ >1/2)?
- Applicable to other deformed NLSM?