nonlinear $\underbrace{\text{Schrödinger}}$ Equation

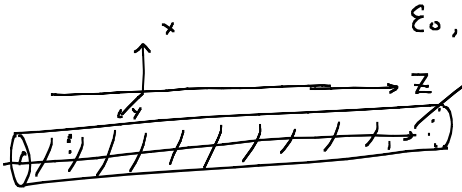
고전. → nonlinear

QM : linear

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec r)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

만약 ψ_1 과 ψ_2 → 해

$|\psi|^2$ 확률 → $c_1\psi_1 + c_2\psi_2$ → 해



ε_0, μ_0

ε, μ

plane polarized $\vec E \propto \hat x$

$$\vec E = \hat x \cdot \frac{1}{2}\Big[\underbrace{\bar E(\vec r, t)\, e^{-i\omega t} + c.c.}_{E_x \to E}\Big]$$

Maxwell Equations : no charge, no current. 진공 아님

$$\vec\nabla\cdot\vec D = 0,\quad \vec\nabla\cdot\vec B = 0 \qquad \vec D = \epsilon\vec E = \epsilon_0\vec E + \vec P$$

$$\vec\nabla\times\vec E = -\frac{\partial\vec B}{\partial t} \qquad \vec B = \mu\vec H$$

$$\vec\nabla\times\vec H = \frac{\partial\vec D}{\partial t}$$

$$\vec\nabla\times\vec\nabla\times\vec E = -\frac{\partial}{\partial t}(\vec\nabla\times\vec B) = -\mu\frac{\partial}{\partial t}\underbrace{(\vec\nabla\times\vec H)}$$

$$\overset{\shortparallel}{-\nabla^2\vec E} \qquad \epsilon\frac{\partial\vec E}{\partial t} = \frac{\partial\vec D}{\partial t}$$

$$\therefore\quad \nabla^2\vec E - \epsilon\mu\frac{\partial^2\vec E}{\partial t^2} = 0$$

$$\left.\begin{aligned}\epsilon_0\mu_0 &= \frac{1}{c^2}\\ \epsilon\mu &= \frac{n^2}{c^2}\end{aligned}\right\} \to \epsilon = \epsilon_0 n^2$$

굴절률 → n

① $n(\omega)$

② $\vec P(\vec E)$

$\vec P(\vec E) = \chi\vec E$ (simple)

$\vec E \to -\vec E$; $\vec P \to -\vec P$

$$P_i = \varepsilon_0\Big(\underset{\text{linear}}{\chi^{(1)}_{ij} E_j} + \cancel{\chi^{(2)}_{ijk} E_j E_k} + \underbrace{\chi^{(3)}_{ijkl} E_j E_k E_l + \cdots}_{\text{nonlinear}}\Big)$$

$$P_x = \varepsilon_0\left(\chi^{(1)}_{xx} E_x + \chi^{(3)}_{xxxx} E_x^3 + \cdots\right)$$

$$D_x = \epsilon E_x = \epsilon_0 E_x + P_x = \underbrace{\varepsilon_0 \overbrace{(1 + \chi^{(1)}_{xx})}^{n^2}}_{\varepsilon} E_x + \varepsilon_0\chi^{(3)}_{xxxx} E_x^3$$

$$\frac{\omega}{k} = \frac{c}{n} \to k = \frac{n\omega}{c} \to k = \underbrace{\frac{n(\omega_0)\omega_0}{c}}_{k_0} + \overbrace{\left.\frac{dk}{d\omega}\right|_{\omega_0}}^{\frac{1}{v_g}}(\omega - \omega_0) + \frac{1}{2}\underbrace{\left.\frac{d^2k}{d\omega^2}\right|_{\omega_0}}_{\equiv k_2}(\omega-\omega_0)^2 + \cdots$$

$$\underbrace{\left(\varepsilon_0 n^2 + \varepsilon_0\chi^{(3)}_{xxxx} E^2\right)}_{\equiv\, \varepsilon_0 \bar n^2} E = \epsilon E$$

$$\to \bar n^2 = n^2 + \chi^{(3)}_{xxxx}|E|^2$$

$$\bar n = n\sqrt{1 + \frac{\chi^{(3)}}{n^2}|E|^2} = n + \underset{\sim\sim}{n_2|E|^2}$$

$$\nabla^2 E - \frac{\bar n^2}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

$$E(\vec r, t) = \int \tilde E(\vec r, \omega)\, e^{-i\omega t}\,\frac{d\omega}{2\pi}$$

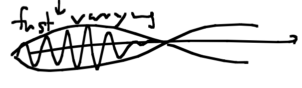
$$\tilde E(\vec r, \omega) = \int E(\vec r, t)\, e^{i\omega t}\, dt$$

$$\nabla^2 \tilde{E} = \int \underbrace{\nabla^2 E}_{\frac{\bar{n}^2}{c^2}\frac{\partial^2 E}{\partial t^2}} e^{i\omega t} = -\frac{n^2\omega^2}{c^2}\tilde{E} + \frac{\chi}{c^2}\int |E|^2 \frac{\partial^2 E}{\partial t^2} e^{i\omega t}$$

$$\bar{n}^2 = n^2 + \chi |E|^2$$

pert. theory : $|\chi| \ll 1$ $\qquad \nabla^2 \tilde{E} + \frac{n^2\omega^2}{c^2}\tilde{E} = 0$

$$\tilde{E}(\vec{r},\omega) = F(x,y)\left[\underbrace{\tilde{A}(z,\omega)}_{\text{slowly varying}} \underbrace{e^{ikz}}_{\text{fast varying}}\right].$$



$$\nabla^2\tilde{E} = (\partial_x^2 + \partial_y^2)F\cdot\tilde{A}e + F\,\partial_z^2[\tilde{A}e] = -\frac{n^2\omega^2}{c^2}F\tilde{A}e$$

$$\partial_z[\tilde{A}e^{ikz}] = \tilde{A}'e + ik\tilde{A}e$$

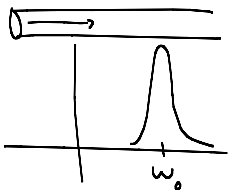
$$\partial_z^2[\quad " \quad] = \underset{\overset{\shortparallel}{0}}{\tilde{A}''}e + 2ik\tilde{A}'e - k^2\tilde{A}e$$

$$\underbrace{\frac{(\partial_x^2+\partial_y^2)F}{F}}_{\alpha}\tilde{A} + 2ik\tilde{A}' - k^2\tilde{A} = -\frac{n^2\omega^2}{c^2}\tilde{A} + \frac{\chi}{c^2}\frac{\overbrace{|E|^2\int\frac{\partial^2 E}{\partial t^2}e^{i\omega t}\,dt}^{-\omega^2\tilde{E} = \cancel{F}\tilde{A}\cancel{e}}}{\cancel{F}\,\cancel{e^{ikz}}}$$

$$\alpha\tilde{A} + 2ik\tilde{A}' = \left(k^2 - \frac{n^2\omega^2}{c^2}\right)\tilde{A} - \frac{\chi^{(3)}}{c^2}\omega^2|E|^2\tilde{A}$$

$$= \left(k - \frac{n\omega}{c}\right)\underbrace{\left(k + \frac{n\omega}{c}\right)}_{\simeq 2k_0}\tilde{A} - \;''$$

$$k(\omega) = k_0 + \frac{1}{v_g}(\omega-\omega_0) + \frac{1}{2}k_2(\omega-\omega_0)^2 + \cdots$$



$k = k_0 = k(\omega)$

$$= 2k_0\,(\underbrace{k_0 - \frac{n\omega}{c}}_{\underbrace{k_0 - \left(\frac{n\omega}{c}\right)_{\omega_0}}_{0}})\tilde{A} - \cdots$$

$$\left(\frac{n\omega}{c}\right)_{\omega_0} = k_0$$

$$\alpha\tilde{A} + 2ik\tilde{A}' = \left[-\frac{2k_0}{v_g}(\omega-\omega_0) - k_2(\omega-\omega_0)^2\right]\tilde{A} - \frac{\chi^{(3)}}{c^2}\omega^2|E|^2\tilde{A}.$$

Inverse FT.

$$\tilde{E} = F\tilde{A}e^{ikz} \qquad \tilde{A}\to A \qquad A = \int \tilde{A}\,e^{i\omega t}\,d\omega$$

$$E = FAe^{ikz}$$

$$\int(\omega-\omega_0)\tilde{A}\,e^{i(\omega-\omega_0)t}\,d\omega = \frac{1}{i}\frac{\partial A}{\partial t}$$

$$-\frac{\partial^2 A}{\partial t^2} = \int(\omega-\omega_0)^2\tilde{A}\,e^{i(\omega-\omega_0)t}\,d\omega$$

$$\alpha A + 2ik_0\frac{\partial A}{\partial z} = i\frac{2k_0}{v_g}\frac{\partial A}{\partial t} + k_2\frac{\partial^2 A}{\partial t^2} - \frac{\chi^{(3)}}{c^2}\omega^2|F|^2|A|^2A$$

$$\underbrace{\frac{\alpha'}{2k_0}}_{\beta}A + i\frac{\partial A}{\partial z} = i\frac{1}{v_g}\frac{\partial A}{\partial t} + \underbrace{\frac{k_2}{2k_0}}_{\alpha}\frac{\partial^2 A}{\partial t^2} - \underbrace{\frac{\chi^{(3)}}{2c^2k_0}\omega^2|F|^2}_{\Gamma}|A|^2A$$

$$\boxed{-i\frac{\partial A}{\partial z} + i\frac{1}{v_g}\frac{\partial A}{\partial t} + \frac{\alpha}{2}\frac{\partial^2 A}{\partial t^2} - \Gamma|A|^2A = \beta A = 0}$$

$\beta = 0$

if $\beta = 0$ if homogeneous in x, y direc.

$\alpha \propto k_2$ nonlinearity $\Gamma \propto \chi^{(3)}$

$\alpha = \left.\dfrac{d^2 k}{d\omega^2}\right|_{\omega_0} \leftarrow$ dispersion. $\Gamma = 0$ if linear

$\alpha = 0$ if dispersionless

① $\alpha = \Gamma = 0$

$$\frac{\partial A}{\partial z} - \frac{1}{v_g}\frac{\partial A}{\partial t} = 0$$

$$0 = \left(\frac{\partial}{\partial z} + \frac{1}{v_g}\frac{\partial}{\partial t}\right)\left[\quad " \quad\right] = \left(\frac{\partial^2}{\partial z^2} - \frac{1}{v_g^2}\frac{\partial^2}{\partial t^2}\right)A = 0$$

$$\frac{1}{v_g} \equiv \left.\frac{\partial k}{\partial \omega}\right|_{\omega_0} = \frac{1}{c}\frac{\partial (n\omega)}{\partial \omega} = \frac{1}{c}\left(n_0'\omega_0 + n_0\right)$$

$n_0 \equiv n(\omega_0)$

$$A = A(z,t) = f(z - v_g t) \qquad k = \frac{n\omega}{c}$$

② $\Gamma = 0$, $\alpha \neq 0$ $\quad A_1(z, \underbrace{t - \tfrac{1}{v_g}z}_{\equiv\, T})$

$$\frac{\partial A_1}{\partial t} = \frac{\partial A}{\partial T}\overbrace{\frac{\partial T}{\partial t}}^{1}$$

$$\frac{\partial A_1}{\partial z} \rightarrow \frac{\partial A}{\partial z} + \frac{\partial A}{\partial T}\overbrace{\frac{\partial T}{\partial z}}^{-\frac{1}{v_g}}$$

$$\underbrace{-i\frac{\partial A}{\partial z} + i\frac{1}{v_g}\frac{\partial A}{\partial t}}\; + \frac{\alpha}{2}\frac{\partial^2 A}{\partial t^2} = 0$$

$$-i\frac{\partial A_*}{\partial z} + \frac{\alpha}{2}\frac{\partial^2 A_*}{\partial T^2} = 0 \qquad A(z,T)$$

linear $\quad A(z,T) = \displaystyle\int \tilde{A}(\omega)\, \underline{e^{i(\omega T + \frac{\alpha}{2}\omega^2 z)}}\, d\omega$

$$\left(-i\frac{\partial}{\partial z} + \frac{\alpha}{2}\frac{\partial^2}{\partial T^2}\right) = \int \tilde{A}(\omega)\Big[\underbrace{+\frac{\alpha}{2}\omega^2 + (i\omega)^2\frac{\alpha}{2}}_{0}\Big]e^{\cdots}\,d\omega = 0$$

③ $\alpha = 0$, $\Gamma \neq 0$

$$A^*\left(i\frac{\partial A}{\partial z} + \Gamma |A|^2 A\right) = 0, \quad A(z,T)$$

$$-)\quad A\left(-i\frac{\partial A^*}{\partial z} + \Gamma |A|^2 A^*\right) = 0$$

$$i\frac{\partial}{\partial z}|A|^2 = 0 \;\rightarrow\; |A|^2 = F(T) = F\left(t - \frac{z}{v_g}\right)$$

$$i\frac{\partial A}{\partial z} + \Gamma F(T) A = 0 \qquad F(T) = |A_0(T)|^2$$

$$\boxed{A = A_0(T)e^{+i\Gamma |A_0(T)|^2 z}}$$

④ $\alpha, \Gamma \neq 0$

Assume $\underline{A = A_0(T)\, e^{i\phi(z)}}$ $\longrightarrow A_0 > 0$

$(A \rightarrow e^{ic}A)$

$$\underbrace{-i\frac{\partial A}{\partial z}} + \frac{\alpha}{2}\frac{\partial^2 A}{\partial T^2} - \Gamma\,\widetilde{|A|^2}A = 0$$

$$\phi' A_0 \cancel{e^{i\phi}} + \frac{\alpha}{2}A_0'' \cancel{e^{i\phi}} - \Gamma |A_0|^2 A_0 \cancel{e^{i\phi}} = 0$$

$$\underset{\frac{d\phi}{dz}(z)}{\phi'} + \underbrace{\frac{\alpha}{2}\frac{A_0''}{A_0} - \Gamma |A_0|^2}_{T \text{ 만의 함수}} = 0$$

상수 g'' $- g$ $= 0$

$$\frac{d\phi}{dz} = g \rightarrow \phi = gz$$

$$\frac{\alpha}{2} A_0'' - \Gamma \underbrace{|A_0|^2 A_0}_{A_0^3} - g A_0 = 0 \qquad A_0 > 0$$

$$\frac{\alpha}{2} A_0' A_0'' - (\Gamma A_0^3 + g A_0) A_0' = 0$$

$$\frac{\alpha}{4} (A_0'^2)' - \left(\frac{\Gamma}{4} A_0^4 + \frac{g}{2} A_0^2 \right)' = 0$$

$$\frac{\alpha}{4} (A_0'^2) - \left(\frac{\Gamma}{4} A_0^4 + \frac{g}{2} A_0^2 \right) = c = 0$$

$$\frac{\alpha}{4} A_0' = \sqrt{\frac{\Gamma}{4} A_0^4 + \frac{g}{2} A_0^2}$$

$$= \sqrt{\frac{g}{2}} A_0 \sqrt{1 + \frac{\Gamma}{2g} A_0^2}$$

$\Gamma < 0$, $\frac{\Gamma}{2g} = -1$ (special sol.)

$$\frac{\alpha}{4} A_0' = \sqrt{\frac{g}{2}} A_0 \sqrt{1 - A_0^2}$$

assume $A_0(T) \equiv \frac{1}{\cosh y} = \operatorname{sech} y = \operatorname{sech} \gamma T$ "

$$A_0' = -\frac{\operatorname{sh} y}{\operatorname{ch}^2 y} y' \qquad -\frac{\alpha}{4} \frac{\cancel{\operatorname{sh} y}}{\operatorname{ch}^2 y} y' = \sqrt{\frac{g}{2}} \frac{1}{\operatorname{ch} y} \frac{\cancel{\operatorname{sh} y}}{\operatorname{ch} y}$$

$$y' \equiv \gamma \rightarrow y = \gamma T$$

$$\gamma = -\sqrt{\frac{g}{2}} \frac{4}{\alpha}$$

$$\rightarrow \underline{A = \operatorname{sech} \gamma T \; e^{igz}} \qquad T = t - \frac{z}{v_g}$$

→ not unique.

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general solution: Zakharov-Shabat solution.

$$\underline{i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial \tau^2} + |u|^2 u = 0}$$

$$+ i \frac{\partial A}{\partial z} - \frac{\alpha}{2} \frac{\partial^2 A}{\partial T^2} + \underline{\Gamma |A|^2 A} = 0 \qquad A = \frac{u}{\sqrt{\Gamma}}$$

$$i \frac{\partial u}{\cancel{\sqrt{\Gamma}} \partial z} - \frac{\alpha}{2 \cancel{\sqrt{\Gamma}}} \frac{\partial^2 u}{\partial T^2} + \overbrace{\frac{|u|^2 u}{\cancel{\sqrt{\Gamma}}}}^{|u|^2 \frac{u}{\sqrt{\Gamma}}} = 0 \rightarrow z \equiv \zeta$$

$$\sqrt{\alpha} T \equiv \tau$$

$$u(z, \tau) = -2 \sum_{j=1}^{N} \lambda_j^* \psi_{2j}^* \qquad \text{"N soliton solution"}$$

$$\lambda_j = \sqrt{c_j} \, e^{(i \zeta_j \tau + i \zeta_j^2 z)}, \qquad c_j = \frac{\prod_{k=1}^{N} (\eta_j + \eta_k)}{\prod_{k(\neq j)=1}^{N} |\eta_j - \eta_k|}$$

$$\zeta_j = \xi_j + i \eta_j$$

$$\psi_{1j} + \sum_{k=1}^{N} \frac{\lambda_j \lambda_k^*}{\zeta_j - \zeta_k^*} \psi_{2k}^* = 0 \qquad j = 1, \cdots, N$$

$$\psi_{2j}^* - \sum_{k=1}^{N} \frac{\lambda_j^* \lambda_k}{\zeta_j^* - \zeta_k} \psi_{1k} = \lambda_j^*$$

N = 1

$$c_1 = 2\eta_1 \quad \rightarrow \quad u = -2\lambda_1^* \psi_{21}^*$$

$$\lambda_1 = \sqrt{c_1}\, e^{(i\zeta_1\tau + i\zeta_1^2 z)} = \sqrt{2\eta_1}\, e^{-\eta_1\tau - i\eta_1^2 z}$$

$$\psi_{11} + \frac{|\lambda_1|^2}{\zeta_1 - \zeta_1^*} \psi_{21}^* = 0$$

$$\psi_{21}^* - \frac{|\lambda_1|^2}{\zeta_1^* - \zeta_1} \underbrace{\psi_{11}}_{-\frac{|\lambda_1|^2}{\zeta_1 - \zeta_1^*}\psi_{21}^*} = \lambda_1^*$$

$$\psi_{21}^* = \frac{\lambda_1^*}{1 + \frac{|\lambda_1|^4}{|\zeta_1 - \zeta_1^*|^2}} \quad \rightarrow u = -2 \frac{2\eta_1 e^{-2\eta_1\tau + 2i\eta_1^2 z}}{1 + \frac{\cancel{(2\eta_1)^2} e^{-4\eta_1\tau}}{\underbrace{|\zeta_1 - \zeta_1^*|^2}_{\cancel{(2\eta_1)^2}}}}$$

$$\zeta_1 = \xi_1 + i\eta_1 = i\eta_1 \quad (\xi_1 = 0)$$

$$u = -2\eta_1 \underbrace{\frac{2}{e^{2\eta_1\tau} + e^{-2\eta_1\tau}}}_{\mathrm{sech}(2\eta_1\tau)} e^{2i\eta_1^2 z}$$

$$\rightarrow \quad u = -2\eta_1\, \mathrm{sech}(2\eta_1\tau)\, e^{2i\eta_1^2 z}$$

(cf) $A = \mathrm{sech}\, \gamma T\, e^{igz}$

$$\gamma = -\sqrt{\frac{g}{2}}\frac{4}{\alpha} \qquad T \equiv \frac{\tau}{\sqrt{-\alpha}}$$

$$\gamma T = +\underbrace{\sqrt{\frac{g}{2}}\frac{4}{|\alpha|}\sqrt{|\alpha|}}_{2\eta_1}\tau \qquad \alpha < 0$$

$$2\eta_1 \rightarrow g = 2\eta_1^2$$

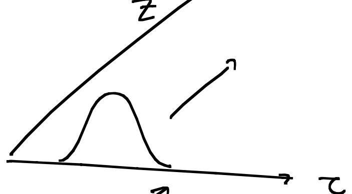
$$|\alpha| = 1 \rightarrow \sqrt{g} =$$

N = 1

N = 2

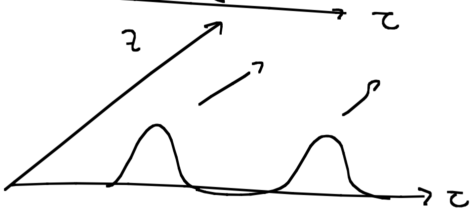
$$\zeta_j = \overset{= 0}{\xi_j} + i\eta_j$$

$$\eta_1 = \frac{1}{2}, \quad \eta_2 = \frac{3}{2} \quad \rightarrow \quad u(z,\tau) = \frac{4\left[\cosh(3\tau) + 3e^{4iz}\cosh\tau\right] e^{i\frac{z}{2}}}{(\mathrm{ch}\,4\tau + 4\,\mathrm{ch}\,2\tau + 3\cos 4z)}$$



soliton interaction

"two soliton"



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