Breather

$$\tan\frac{\phi}{4} = \frac{1}{v}\,\frac{\operatorname{sh}\left[\left(\frac{1}{a}-a\right)\frac{t}{2}\right]}{\operatorname{ch}\left[\left(a+\frac{1}{a}\right)\frac{x}{2}\right]} = \frac{1}{v}\,\frac{\operatorname{sh}(\gamma v t)}{\operatorname{ch}(\gamma x)} = \frac{1}{Iu}\,\frac{\operatorname{sh}(I\gamma_u u t)}{\cosh(\gamma_u x)} = \frac{\sin(\gamma_u u t)}{u\,\operatorname{ch}(\gamma_u x)}$$

let $v = Iu = \dfrac{\frac{1}{a}-a}{\frac{1}{a}+a}$

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Now, we look for periodic soliton.

$$\phi(x,t) = \phi(x+L,t)$$

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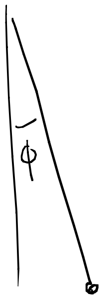


$$E = \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi = -mgl\cos\phi_0$$

$$t = 4\sqrt{\frac{l}{2g}}\int_0^{\phi}\frac{d\phi}{\sqrt{\cos\phi-\cos\phi_0}}$$

$$T = 4\sqrt{\frac{l}{2g}}\int_0^{\phi_0}\frac{d\phi}{\sqrt{\cos\phi-\cos\phi_0}}$$

$$= 2\sqrt{\frac{l}{g}}\int_0^{\phi_0}\frac{d\phi}{\sqrt{\sin^2\frac{\phi_0}{2}-\sin^2\frac{\phi}{2}}} = \cdots\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k^2\sin^2\frac{\phi}{2}}}$$

$$\sin\xi \equiv \frac{\sin\frac{\phi}{2}}{\sin\frac{\phi_0}{2}}$$

# Elliptic functions

let $u = \int_0^{\phi} \frac{d\alpha}{\sqrt{1 - m\sin^2\alpha}} = F(\phi, m) \Rightarrow \phi = am(u|m)$ "amplitude"

elliptic integral

$$sn\, u \equiv \sin(\phi) = \sin(am(u))$$

$$cn\, u \equiv \cos(\phi) = \cos(am(u))$$

$$dn\, u \equiv \sqrt{1 - m\sin^2\phi} = \left(\frac{dF}{d\phi}\right)^{-1} = \frac{d\phi}{du}$$

$$\therefore \quad \phi' = \sqrt{1 - m\frac{1-\cos 2\phi}{2}} = \sqrt{\left(1 - \frac{m}{2}\right) + \frac{m}{2}\cos 2\phi}$$

$am(u|m)$

$$(2\phi)' = \sqrt{(4 - 2m) + 2m\cos 2\phi}$$

$$2\phi = \psi - \pi \qquad \psi' = \sqrt{(4-2m) - 2m\cos\psi}$$

---

SG Eq: $\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi + \sin\phi = 0$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - (1 - \cos\phi) \rightarrow \phi = \phi(\underbrace{kx - \omega t}_{u})$$

$$(\partial_\mu\phi)^2 + \sin\phi = 0 \longrightarrow (\omega^2 - k^2)\phi'' + \sin\phi = 0$$

multiply $\phi'$ & int.

$$(\omega^2 - k^2)\frac{1}{2}\phi' - \cos\phi = C \qquad (\text{as } u \to \pm\infty)$$

$$\therefore \quad \phi'_{SG} = \sqrt{\frac{2}{k^2 - \omega^2}(C - \cos\phi)}$$

$$m = \frac{1}{k^2 - \omega^2} \qquad cm = 2 - m$$

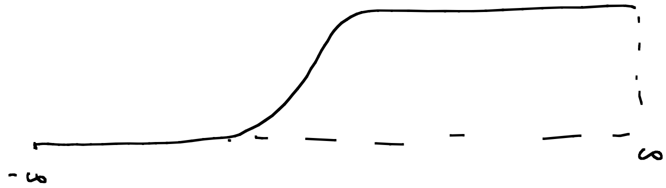
$$\therefore \qquad \left(C = \frac{2}{m} - 1\right)$$

$$\therefore \quad \boxed{\phi_{SG} = \psi = \pi + 2\, am\left(kx - \omega t \,\Big|\, \frac{1}{k^2 - \omega^2}\right)}$$

periodicity

$$am(u|m) = am(u + 2K(m)|m)$$

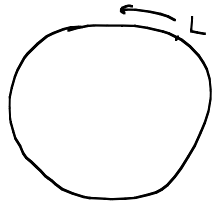
$$\therefore \quad x \longrightarrow \frac{2K(m)}{k} \text{ is periodicity}$$




$-\infty \le x \le \infty$

$L \to \infty$

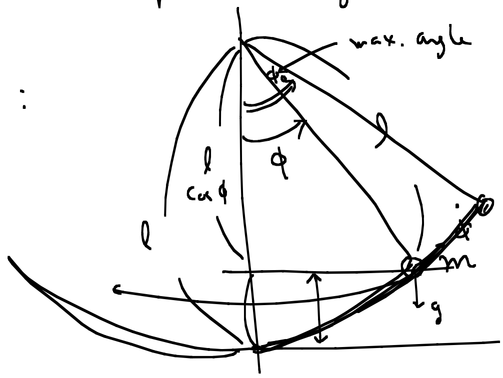
periodic soliton : $\phi(x,t) = \phi(x+L,t)$




finite-size 공간의 soliton

## 타원함수 elliptic integrals

(ex) 단진자 :




$$E = \frac{1}{2} m (l\dot{\phi})^2 + \underline{l(1-\cos\phi)mg}$$

$\phi \le \phi_0 \ll 1$

$1-\cos\phi \approx \frac{\phi^2}{2}$

$\phi_0 \sim O(1)$

$$= \underline{l(1-\cos\phi_0)mg}$$

$T$

$$\dot{\phi}^2 = \frac{\cancel{l}\cancel{m}g}{\frac{\cancel{m}l^{\cancel{2}}}{2}}(\cos\phi - \cos\phi_0) \rightarrow \left(\frac{d\phi}{dt}\right)^2 = \frac{2g}{l}\left(\underset{\cancel{1}-2\sin^2\frac{\phi}{2}}{\cos\phi} - \underbrace{\cos\phi_0}_{\cancel{1}-2\sin^2\frac{\phi_0}{2}}\right)$$

$$= \frac{4g}{l}\left(\sin^2\frac{\phi_0}{2} - \sin^2\frac{\phi}{2}\right)$$

$$\int_0^\phi \frac{d\phi}{\sqrt{\sin^2\frac{\phi_0}{2} - \sin^2\frac{\phi}{2}}} = \sqrt{\frac{4g}{l}}\int_0^t dt$$

$$\frac{\sin\frac{\phi}{2}}{\sin\frac{\phi_0}{2}} = \sin\alpha \rightarrow \frac{\frac{1}{2}\cos\frac{\phi}{2}}{\sin\frac{\phi_0}{2}}d\phi = \cos\alpha\, d\alpha$$

$$= \int_0^\alpha \frac{2\cancel{\sin\frac{\phi_0}{2}}\cancel{\cos\alpha}\, d\alpha}{\cos\frac{\phi}{2}\sqrt{\sin^2\frac{\phi_0}{2} - \sin^2\frac{\phi_0}{2}\sin^2\alpha}} \quad \cancel{\sin\frac{\phi_0}{2}}\cancel{\cos\alpha} = 2\underbrace{\int_0^\alpha \frac{d\alpha}{\sqrt{1-\underbrace{\sin^2\frac{\phi_0}{2}}_{k^2}\sin^2\alpha}}}$$

$$\cancel{2}\underbrace{\int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}} = \cancel{2}\sqrt{\frac{g}{l}}\,\frac{T}{4} \rightarrow \boxed{T = 4\sqrt{\frac{l}{g}}\,K\left(\sin^2\frac{\phi_0}{2}\right)}$$

complete elliptic integral of 1st kind $K(k^2)$

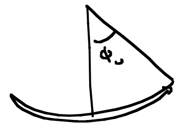
$$\underbrace{\int_0^\alpha \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}} = \sqrt{\frac{g}{l}}\, t \qquad \sin\alpha = \frac{\sin\frac{\phi}{2}}{\sin\frac{\phi_0}{2}} \qquad k^2 = \sin^2\frac{\phi_0}{2}$$

$$\underline{F(\alpha, k^2) = \sqrt{\frac{g}{l}}\, t} \longrightarrow \phi = \phi(t)$$

in "

if $\phi_0 \ll 1$

$$T = 4\sqrt{\frac{l}{g}}\,\underline{K\left(\sin^2\frac{\phi_0}{2}\right)} = 4\sqrt{\frac{l}{g}}\left(\frac{\pi}{2} + \frac{\pi}{8}\cdot\sin^2\frac{\phi_0}{2} + \cdots\right)$$



$$T = 2\pi\sqrt{\frac{l}{g}} + \frac{\pi}{2}\sqrt{\frac{l}{g}}\sin^2\frac{\phi_0}{2} + \cdots$$

---

$$\frac{du}{d\phi} = \frac{1}{\sqrt{1-m\sin^2\phi}} \qquad F(\phi, m) = \int_0^{\phi}\frac{d\phi}{\sqrt{1-m\sin^2\phi}} = u \qquad F(\phi, m) = u$$

$$\boxed{\sqrt{1-m\sin^2\phi} = \frac{d\phi}{du}} \to \phi = \mathrm{am}(u, m) \qquad \sin\alpha = \frac{\sin\frac{\phi}{2}}{\sin\frac{\phi_0}{2}} \qquad \phi = \mathrm{am}(u, m)$$

Jacobi amplitude

$$F(\alpha, m) = \sqrt{\frac{g}{l}}\, t \Rightarrow \alpha = \mathrm{am}\left(\sqrt{\frac{g}{l}}\, t, \sin^2\frac{\phi_0}{2}\right)$$

$$\sin\frac{\phi}{2} = \sin\frac{\phi_0}{2}\,\sin\left(\mathrm{am}\left(\sqrt{\frac{g}{l}}\, t, \sin^2\frac{\phi_0}{2}\right)\right)$$

$$\phi(t) = 2\sin^{-1}\left[\sin\frac{\phi_0}{2}\,\sin\left(\mathrm{am}\left(\sqrt{\frac{g}{l}}\, t, \sin^2\frac{\phi_0}{2}\right)\right)\right] = 2\sin^{-1}\left[\sin\frac{\phi_0}{2}\,\mathrm{sn}\left(\sqrt{\frac{g}{l}}\, t \,\middle|\, \sin^2\frac{\phi_0}{2}\right)\right]$$

elliptic modulus

$$\sin(\mathrm{am}(u, m)) = \mathrm{sn}(u \mid m)$$

Jacobi SN function

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$$\partial_\mu\partial^\mu\phi = -\sin\phi = \ddot{\phi} - \phi'' \qquad \mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - (1-\cos\phi)$$

static

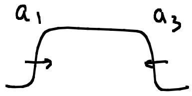
$$\phi'\phi'' = +\sin\phi\,\phi'$$

$$\left(\frac{\phi'^2}{2}\right)' = -(\cos\phi)' \to \boxed{\phi'^2 = -2\cos\phi + C}$$

$$\phi' = \sqrt{-2\cos\phi + C}$$

$$x \to \pm\infty \qquad 0 = \sqrt{2 - 2\cos\phi} = 2\sin\frac{\phi}{2}$$

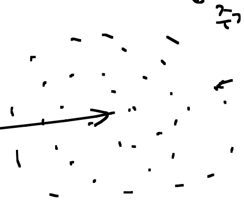
$$\phi = 4\tan^{-1}(e^{y}) \qquad y = \gamma(x - vt)$$



$a_1 \quad a_3$

$$a_1 = \frac{1}{a_3};\ \phi = 4\tan^{-1}\left(\frac{1}{v}\,\frac{\mathrm{sh}(\gamma v t)}{\mathrm{ch}(\gamma x)}\right) \qquad \boxed{x \to x + L}$$

$$\phi = 4\tan^{-1}\left(g(t)\underbrace{f(x)}_{\text{주기함수}}\right)$$



← two level :

$n \Rightarrow \cos\sigma \to$ SG eq.

(유한 길이 공간 안의 솔리톤)

finite-size soliton.

① kink train ② Fluxon ③ Fluxon Breather ④ Plasmon

$sn(u,m) = \sin(am(u,m))$

$cn(u,m) = \sqrt{1 - sn^2(u,m)}$

$dn(u,m)$

$\boxed{\sqrt{1-m\sin^2\Phi} = \frac{d\Phi}{du}} \rightarrow \boxed{\Phi = am(u,m)}$

$\underline{m=1}$ . $\phi = \phi(y)$

$\uparrow$ $y = \gamma(x - vt)$

$-\sin\phi = \ddot{\phi} - \phi''$ $\leftarrow$ $\phi = \phi(\underbrace{kx - \omega t}_{u})$ $k = \gamma$, $\omega = kv$

$-\sin(\phi(u))\phi' = (\omega^2 - k^2)\phi''(u)\ \phi'$

$\cos(\phi) = \frac{1}{2}(\omega^2 - k^2)\phi'^2 + c$

$\frac{d\phi}{du} = \sqrt{(\cos\phi - c)\frac{2}{\omega^2 - k^2}}$

$\boxed{\phi \equiv \pi + 2\Phi} \rightarrow \cos\phi = -\cos 2\Phi$

$= -(1 - 2\sin^2\Phi)$

$2\frac{d\Phi}{du} = \sqrt{\frac{2}{k^2 - \omega^2}(c-1) - \frac{4}{k^2 - \omega^2}\sin^2\Phi}$

$\frac{d\Phi}{du} = \sqrt{\frac{1/2}{k^2 - \omega^2}(c-1) - \underbrace{\frac{1}{k^2 - \omega^2}}_{m}\sin^2\Phi}$

(cf) $\boxed{\sqrt{1-m\sin^2\Phi} = \frac{d\Phi}{du}} \rightarrow \boxed{\Phi = am(u,m)}$ $m = \frac{1}{k^2 - \omega^2}$

$\frac{m}{2}(c-1) = 1$ $\underline{c = 1 + \frac{2}{m}}$

$\therefore \Phi = am(u, m)$

$\therefore$ $\boxed{\phi(x,t) = \pi + 2\,am\left(\overbrace{\underbrace{kx - \omega t}_{\downarrow\ u + 4K(m)}}^{u}, \underbrace{\frac{1}{k^2 - \omega^2}}_{m}\right)}$

* $kx - \omega t = y = \gamma x - \gamma v t$ $\gamma = \frac{1}{\sqrt{1-v^2}} = k$ $\omega = \gamma v$

$k^2 - \omega^2 = 1 \rightarrow m = 1.$

$\phi(x+L, t) = \phi(x,t) \leftarrow L = \frac{4K(m)}{k}$ $K(m) \rightarrow a - \ln(m-1)$, $(m-1) \rightarrow L$

$\phi(x, t+T) = \phi(x,t)$ $T = \frac{4K(m)}{\omega}$

$m \sim 1$ $\underline{\phi(y,t) = 4\tan^{-1}(e^y)} + [\ ?\ ]\ \underbrace{e^{-\frac{kL}{4}}}_{\sim m-1}$ $(m-1)$

$L = \frac{4}{k}(a - \ln(m-1))$

$\downarrow$ $m - 1 \simeq e^{-\frac{k}{4}(L-a)}$

$\phi(x,t) = \underline{4\tan^{-1}(e^y)} + \frac{1}{2}\left(\frac{kx - \omega t}{ch(kx - \omega t)} - sh(kx - \omega t)\right)\underbrace{e^{-\frac{k}{4}(L-a)}}_{L \gg 1}$

$\longleftrightarrow$ finite-size effect.

Jacobi DN

**Fluxon.** $\phi(x,t) = 4\arctan\left[\frac{k}{\omega}\,dn(kx, m_x)\,sc(\omega t, m_t)\right]$

JacobiSC

$$m_x = 1 - \frac{\omega^2}{k^2}\,\frac{1-k^2+\omega^2}{\omega^2-k^2}$$

$$m_t = 1 - \frac{k^2}{\omega^2}\,\frac{1-k^2+\omega^2}{\omega^2-k^2}$$

H.W.1 $m_x, m_t \to 1$ limit을 고려하여 finite-size effect를 구하라

**Breather** $\phi(x,t) = 4\arctan\left[\frac{k}{\omega}\,dn(kx, m_x)\,sn(\omega t, m_t)\right]$

$$m_x = 1 - \frac{\omega^2}{k^2}\,\frac{1-k^2-\omega^2}{\omega^2+k^2}$$

$$m_t = \frac{k^2}{\omega^2}\,\frac{1-k^2-\omega^2}{\omega^2+k^2}$$

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**plasmon** $\phi(x,t) = 4\tan^{-1}\left[A\,cn(kx, m_x)\,cn(\omega t, m_t)\right]$

$$A = \sqrt{\frac{1+k^2-\omega^2}{1-k^2+\omega^2}}$$

$$m_x = \frac{(1+k^2)^2-\omega^4}{4k^2}$$

$$m_t = \frac{k^4-(1-\omega^2)^2}{4\omega^2}$$

$\omega^2 \to k^2-1 \;(k^2 \to \omega^2+1)\; m_x \to 1,\; m_t \to 1.$

H.W.2 $\omega^2 \simeq k^2-1$ 일때
$m_x, m_t \to 1$ limit을 고려하여 finite-size effect를 구하라