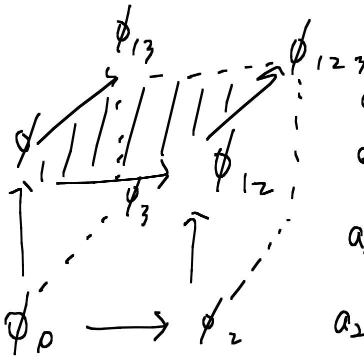
$$\partial_+\left(\frac{\phi_1-\phi_{12}}{2}\right) = a_2 \sin\left(\frac{\phi_1+\phi_{12}}{2}\right) \cdots ①$$

$$\partial_+\left(\frac{\phi_1-\phi_{13}}{2}\right) = a_3 \sin\left(\frac{\phi_1+\phi_{13}}{2}\right) \cdots ②$$

$$\partial_+\left(\frac{\phi_{12}-\phi_{123}}{2}\right) = a_3 \sin\left(\frac{\phi_{12}+\phi_{123}}{2}\right) \cdots ③$$

$$\partial_+\left(\frac{\phi_{13}-\phi_{123}}{2}\right) = a_2 \sin\left(\frac{\phi_{13}+\phi_{123}}{2}\right) \cdots ④$$



$$① - ④ : \ \partial_+\left(\frac{\phi_1-\phi_{12}-\phi_{13}+\phi_{123}}{2}\right) = 2a_2 \cos\left(\frac{\phi_1+\phi_{12}+\phi_{13}+\phi_{123}}{4}\right)\sin\left(\frac{\phi_1+\phi_{12}-\phi_{13}-\phi_{123}}{4}\right)$$

$$② - ③ : \ \partial_+\left(\frac{\phi_1-\phi_{13}-\phi_{12}+\phi_{123}}{2}\right) = 2a_3 \cos\left(\frac{\phi_1+\phi_{12}+\phi_{13}+\phi_{123}}{4}\right)\sin\left(\frac{\phi_1+\phi_{13}-\phi_{12}-\phi_{123}}{4}\right)$$

$$a_2 \sin\left(\frac{\phi_1+\phi_{12}-\phi_{13}-\phi_{123}}{4}\right) = a_3 \sin\left(\frac{\phi_1+\phi_{13}-\phi_{12}-\phi_{123}}{4}\right)$$

α β

$$a_2(\sin\alpha\cos\beta+\cos\alpha\sin\beta) = a_3(\sin\alpha\cos\beta-\cos\alpha\sin\beta)$$

$$(a_2-a_3)\sin\alpha\cos\beta = -(a_2+a_3)\cos\alpha\sin\beta$$

$$\tan\left(\frac{\phi_{123}-\phi_1}{4}\right) = -\tan\alpha = +\frac{a_2+a_3}{a_2-a_3}\tan\beta = \frac{a_2+a_3}{a_2-a_3}\tan\left(\frac{\phi_{12}-\phi_{13}}{4}\right) \quad : \quad \tan\left(\frac{\phi_{12}-\phi_0}{4}\right)$$

$$\frac{\tan\frac{\phi_{123}}{4}-\tan\frac{\phi_1}{4}}{1+\tan\frac{\phi_{123}}{4}\tan\frac{\phi_1}{4}} = \frac{a_2+a_3}{a_2-a_3}\,\frac{\tan\frac{\phi_{12}}{4}-\tan\frac{\phi_{13}}{4}}{1+\tan\frac{\phi_{12}}{4}\tan\frac{\phi_{13}}{4}}$$

$$\left(\begin{aligned}\tan\frac{\phi_{12}}{4} &= \frac{a_1+a_2}{a_1-a_2}\,\frac{\sinh\frac{y_1-y_2}{2}}{\cosh\frac{y_1+y_2}{2}} = \frac{a_1+a_2}{a_1-a_2}\,\mathrm{th}\frac{y_1+y_2}{4}\\ \tan\frac{\phi_{13}}{4} &= \frac{a_1+a_3}{a_1-a_3}\,\frac{\sinh\frac{y_1-y_3}{2}}{\cosh\frac{y_1+y_3}{2}}\end{aligned}\right.$$

$$\left(\tan\frac{\phi_{123}}{4}-\tan\frac{\phi_1}{4}\right)(a_2-a_3) = (a_2+a_3)\left(\frac{\tan\frac{\phi_{12}}{4}-\tan\frac{\phi_{13}}{4}}{1+\tan\frac{\phi_{12}}{4}\tan\frac{\phi_{13}}{4}}\right)\left(1+\tan\frac{\phi_{123}}{4}\tan\frac{\phi_1}{4}\right)$$

$$\tan\frac{\phi_{123}}{4}\left\{(a_2-a_3)-(a_2+a_3)\left(\frac{\tan\frac{\phi_{12}}{4}-\tan\frac{\phi_{13}}{4}}{1+\tan\frac{\phi_{12}}{4}\tan\frac{\phi_{13}}{4}}\right)\tan\frac{\phi_1}{4}\right\} = (a_2+a_3)\left(\frac{\tan\frac{\phi_{12}}{4}-\tan\frac{\phi_{13}}{4}}{1+\tan\frac{\phi_{12}}{4}\tan\frac{\phi_{13}}{4}}\right) + (a_2-a_3)\tan\frac{\phi_1}{4}$$

$$\tan\frac{\phi_{123}}{4} = \frac{(a_2+a_3)\left(\tan\frac{\phi_{12}}{4}-\tan\frac{\phi_{13}}{4}\right)+(a_2-a_3)\tan\frac{\phi_1}{4}\left(1+\tan\frac{\phi_{12}}{4}\tan\frac{\phi_{13}}{4}\right)}{(a_2-a_3)\left(1+\tan\frac{\phi_{12}}{4}\tan\frac{\phi_{13}}{4}\right)-(a_2+a_3)\left(\tan\frac{\phi_{12}}{4}-\tan\frac{\phi_{13}}{4}\right)}$$

$$= \frac{(a_2+a_3)\left(\frac{a_1+a_2}{a_1-a_2}\frac{\sinh\frac{y_1-y_2}{2}}{\cosh\frac{y_1+y_2}{2}}-\frac{a_1+a_3}{a_1-a_3}\frac{\sinh\frac{y_1-y_3}{2}}{\cosh\frac{y_1+y_3}{2}}\right)+(a_2-a_3)e^{y_1}\left(1+\frac{(a_1+a_2)(a_1+a_3)\sinh\frac{y_1-y_3}{2}\sinh\frac{y_1-y_2}{2}}{(a_1-a_2)(a_1-a_3)\cosh\frac{y_1-y_3}{2}\cosh\frac{y_1-y_2}{2}}\right)}{(a_2-a_3)\left(1+\frac{(a_1+a_2)(a_1+a_3)}{(a_1-a_2)(a_1-a_3)}\frac{\sinh\frac{y_1-y_2}{2}\sinh\frac{y_1-y_3}{2}}{\cosh\frac{y_1-y_3}{2}\cosh\frac{y_1-y_2}{2}}\right)-(a_2+a_3)\left(\frac{(a_1+a_2)\sinh\frac{y_1+y}{2}}{(a_1-a_2)\cosh\frac{y_1-y_2}{2}}-\frac{a_1+a_3}{a_1-a_3}\frac{\sinh\frac{y_1-y_3}{2}}{\cosh\frac{y_1-y_3}{2}}\right)}$$

$$\tan\left(\frac{\phi_{1\cdots n}-\phi_{1\cdots n-2}}{4}\right) = \frac{a_{n-1}+a_n}{a_{n-1}-a_n}$$

$$\begin{tikzcd} \phi_{1\cdots n,n+2} \arrow[r, "a_{n+1}"] & \phi_{1\cdots n+2} \\ \phi_{1\cdots n} \arrow[r, "a_{n+1}"'] \arrow[u, "a_{n+2}"] & \phi_{1\cdots n+1} \arrow[u, "a_{n+2}"'] \end{tikzcd}$$

$$\boxed{\tan\frac{\phi_{1\cdots(n+2)}-\phi_{1\cdots n}}{4} = \frac{a_{n+1}+a_{n+2}}{a_{n+1}-a_{n+2}}\tan\left(\frac{\phi_{1\cdots n+1}-\phi_{1\cdots n,n+2}}{4}\right)}$$

two-soliton

$$\tan\frac{\phi_{13}}{4} = \frac{a_1+a_3}{a_1-a_3}\,\frac{\sinh\frac{y_1-y_3}{2}}{\cosh\frac{y_1+y_3}{2}}$$

$$y_i = \underbrace{\frac{a_i+\frac{1}{a_i}}{2}}_{\gamma_i = \frac{1}{\sqrt{1-v_i^2}}}(x-x_0) + \underbrace{\frac{a_i-\frac{1}{a_i}}{2}}_{-\gamma_i v_i}\, t$$

$$= \gamma_i\,(x - x_0 - \underline{v_i}\, t)$$

$0 < a < 1 \;:\; v > 0$

$a > 1 \;:\; v < 0$

$a = 1 \;:\; v = 0$

$a < 0 \;;\; a = -|a| \qquad \gamma \to -\gamma$

sol $\leftrightarrow$ antisol.

a : complex

$$\frac{\operatorname{sh}\frac{y_1-y_3}{2}}{\operatorname{ch}\frac{y_1+y_3}{2}} = \frac{\operatorname{sh}\frac{1}{4}\left((a_1+\frac{1}{a_1})x + (a_1-\frac{1}{a_1})t - (a_3+\frac{1}{a_3})x - (a_3-\frac{1}{a_3})t\right)}{\operatorname{ch}\frac{1}{4}\left((a_1+\frac{1}{a_1})x + (a_1-\frac{1}{a_1})t + (a_3+\frac{1}{a_3})x + (a_3-\frac{1}{a_3})t\right)}$$

$$a_1+\frac{1}{a_1} = a_3+\frac{1}{a_3} \qquad a_1-\frac{1}{a_1} \neq a_3-\frac{1}{a_3}$$

$a_1+\frac{1}{a_1} = \alpha$

$a_1-\frac{1}{a_1} = i\beta$

$a_1 = e^{i\theta}$

$$\boxed{a_1 = \frac{1}{a_3}}$$

$$= \frac{\operatorname{sh}\frac{1}{2}\overbrace{(a_1-\frac{1}{a_1})}^{\text{imaginary}}\,t}{\operatorname{ch}\frac{1}{2}(a_1+\frac{1}{a_1})\,x}$$

$\alpha = 2\cos\theta$

$\beta = 2\sin\theta$

$$= \frac{\cancel{i}\sin(\sin\theta\, t)}{\operatorname{ch}(\cos\theta\, x)}$$

$$\tan\frac{\phi_{13}}{4} = \frac{a_1+\frac{1}{a_1}}{a_1-\frac{1}{a_1}}\,\frac{\sinh\frac{y_1-y_3}{2}}{\cosh\frac{y_1+y_3}{2}}$$

$\cancel{2}\cos\theta$

$\cancel{2}\cancel{i}\sin\theta$

$\operatorname{ch}\beta\, t - \operatorname{sh}\beta\, t$

주기

$\sin\theta\, t = 2\pi$

$t = \frac{2\pi}{\sin\theta}$

$$\tan\frac{\phi_{13}}{4} = \cot\theta\,\frac{\sin(\sin\theta\, t)}{\operatorname{ch}(\cos\theta\,(x-x_0))}$$

$\operatorname{ch}\beta\, x \pm \operatorname{sh}\beta\, t$

↳ Breather solution.

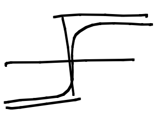
(S-A)

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \operatorname{ch}\beta & +\operatorname{sh}\beta \\ \operatorname{sh}\beta & \operatorname{ch}\beta \end{pmatrix}\begin{pmatrix} t' \\ x' \end{pmatrix}$$

$\operatorname{ch}u$

$-1 \le \operatorname{th}u = v \le 1$

$$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}$$



$t = t'\operatorname{ch}u - x'\operatorname{sh}u$

$x = -t'\operatorname{sh}u + x'\cosh u$

$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \operatorname{ch}u$

$1 = \operatorname{ch}^2u\,(1-\beta^2)$

$\beta^2 = 1 - \frac{1}{\operatorname{ch}^2u} = \operatorname{th}^2u$

$H = p\dot{x} - L \qquad p = \frac{\partial L}{\partial \dot{x}}$

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \boxed{\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + (1-\cos\phi)}$$

$\pi = \dot{\phi}$

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{\partial\mathcal{L}}{\partial\phi} \qquad \mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - (1-\cos\phi)$$

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$$

$$\mathcal{L} = \frac{1}{2}(\dot{\phi}^2 - \phi'^2) - (1-\cos\phi)$$

$$\partial_\mu^2\phi = \underline{\ddot{\phi} - \phi'' + \sin\phi = 0}$$